THE GEOMETRY OF BINOCULAR SPACE PERCEPTION

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FOREWORD

We think that this is an important report because here, for the first time, extensive experimental results and analytical details are presented which strongly support the Luneburg Theory of the Geometry of Binocular Visual Space.

This is a terminal report on research done under contract with the Office of Naval Research (N6onr27119; NR 143-638). The work is being terminated because of our inability to acquire and retain adequate personnel with the highly technical skills necessary for such work. A very high degree of mathematical analytical ability must be in constant and harmonious rapport with an equally high degree of laboratory experimental skills in order to carry out these investigations. In the untimely death of Rudolph Luneburg we suffered an extremely severe loss. After a lapse of two years we were fortunate in acquiring through Professor Richard Courant one of Dr. Luneburg's associates, Dr. Albert A. Blank, who has shown brilliance in his mathematical attack. All the new mathematical analysis herein described and most of the formulation of this report are due to his efforts.

Our mathematical consultant, Dr. Paul Boeder, has given much time and enthusiastic encouragement to our working staff. Professor H.S.M. Coxeter, as a specialist in the non-euclidean geometries, contributed important suggestions which were partly carried out in the ancillary investigations of Dr. Charles Campbell who earned the D. Sc. degree for his part in this research. Dr. Bernard Altschuler and Dr. Anna Stein spent respectively one year and two years in the mathematical analyses during the early part of the study. The largest part of the actual experimentation was carried out by Dr. Gertrude Rand and Miss M. Catherine Rittler.

LeGrand H. Hardy
Principal Investigator

PREFACE

This is a report of progress, theoretical and experimental, in the study of binocular space perception based on the theory of R. K. Luneburg.[1,2,3] The experimental evidence definitely supports Luneburg's major conclusion that the darkroom visual space has a determinate non-euclidean metric or psychometric distance function which is a personal characteristic of the observer. In this report the metric has been developed in terms of coordinates closely related to, but different from those of Luneburg. Much the same methods are used for determining the form of the metric as were suggested by Luneburg. The theory gives an explanation of several well-known perceptual space phenomena such as the frontal geodesics, Blumenfeld alleys, and size constancy.

We have not attempted here to present a review of all our work of the past five years, but only that portion of it which still appears relevant and cogent. It would be fruitless to describe all the false clues and blind alleys that as a rule accompany the formation of any new theory. On the other hand, we are conscious that there are many gaps in our testing program. We employed a very limited number of observers because so few were available for experimentation extending over so long a period. We did not investigate every open door because so many doors were open.

The greatest setback to our research was the untimely loss of our beloved friend and colleague Dr. Rudolph K. Luneburg. To Luneburg we owe the basic concepts and formulation of the theory. His was the guiding hand for more than half of our experimental work. To him we dedicate this paper and hope that this work may stand in his name.

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PART I

THE MATHEMATICAL BASIS OF THE LUNEBURG THEORY

1. INTRODUCTION *

The qualities of form and localization are the basic materials of geometry and in formulating a theory of binocular space perception we attempt to establish the relations which exist between the perception of these qualities and the objective forms and localizations of the physical world.

This study seeks to demonstrate a correlation between the geometrical stimulus presented to an observer and the *geometrically* relevant part of the observer's response. Such visual phenomena as color and brightness, for example, are not considered here.

This discussion could be phrased in operational language, say in terms of "stimulus" and "response" or "input" and "output". The "stimulus" or "input" consists of a physical situation to which the observer is exposed together with a set of instructions; the "response" or "output" is the consequent modification of the physical situation by the observer together with his relevant statements. The use here of terms such as the nouns: perception, appearance, impression; the verbs: to perceive, to sense, to appear; the adjectives: perceptual, subjective, sensed, perceived; and other terms of the same kind, may be considered operationally as a reference to their undefined use in the instructions or in the statement of the observer. For example, the statement that the observer "perceives" the point P_o to be midway between P_1 and P_2 on a straight line may be interpreted in either of two operational meanings: (a) the observer says, "I have the impression that P_o is midway between P_1 and P_2"; or (b) in response to the instruction, "Adjust the position of the light P_o until it appears to be midway between P_1 and P_2 on a straight line", the observer has placed a given physical light in a particular position.

With this understanding, we shall freely employ the words "impression", "perception", etc. in this intuitive way without further clarification.

We are concerned here only with one type of visual stimulus, important insofar as geometrical properties are concerned. This stimulus is characterized as a

* FOR HELPING US TO CLARIFY THE IDEAS OF THIS SECTION, ALTHOUGH ONLY WE ARE RESPONSIBLE FOR THE FORM IN WHICH THEY ARE PRESENTED HERE, WE OWE THANKS TO PROF. C. H. GRAHAM OF COLUMBIA UNIVERSITY AND THE LATE DR. W. BERRY OF ONR.

distribution of luminous points in definite orientations and localizations with respect to the observer. In particular, for our study of binocular space perception, a stimulus is considered to be defined if we give the position of the observer and the positions of the points which are effective in the stimulation of both eyes. Such a stimulus can be characterized numerically in terms of a suitable coordinate system. The *total stimulus configuration* is the set of all points which are binocularly effective. Since this terminology is unwieldy we shall often refer simply to the *stimulus configuration* or the *stimulus*. It is to be understood, however, that both of these terms, wherever they occur, are meant to include *all* effective points.

To measure his spatial perception, the observer is asked to modify an initial stimulus configuration so as to give himself a specified desired impression. The instruction is generally a request to set up a sensory situation which lends itself to description by a mathematical relation of equality. For example, the observer may be presented with three light points Q_1, Q_2, Q_3 arbitrarily located in his binocular field. He is then asked, without moving points Q_1 and Q_3, to adjust the position of the point Q_2 so as to give himself the impression of points P_1, P_2, P_3 placed in that order on a straight line.* This perception of straightness and order is described by the equation

$$(P_1, P_2) + (P_2, P_3) = (P_1, P_3)$$

where, in general, for any pair of points P_i, P_j, the symbol (P_i, P_j) denotes the sensed distance between the points P_i and P_j.

By employing a sufficient variety and number of specific initial conditions upon constructions of diverse kinds we may hope to establish statistically a functional** dependence of perception upon stimulus which may be considered a constant characteristic of the observer. In this way, given the mathematical description of the stimulus, it is possible to describe some constants of the observer's visual responses; that is, to give a mathematical description of the impressions of localization and form with respect to the observer's personal mental frame of reference – the observer's *visual space*.

* WE SHALL ADHERE THROUGHOUT TO THE CONVENTION OF DESIGNATING A PHYSICAL POINT BY Q, THE PERCEIVED POINT BY P.

** THE WORD FUNCTION IS USED HERE IN THE MATHEMATICIAN'S SENSE AND IT MAY BE WELL TO REPEAT THE DEFINITION FOR THE NON-MATHEMATICAL READER:

LET S AND T DENOTE TWO AGGREGATES (OR SETS OR CLASSES) CONSISTING OF ANY ELEMENTS WHATEVER. A (SINGLE-VALUED) FUNCTION DEFINED UPON THE SET S WITH VALUES IN THE SET T IS A MEANS OF ASSOCIATING WITH EACH ELEMENT OF S A UNIQUE ELEMENT OF T. WE ALSO SAY THAT S IS MAPPED INTO T. A FUNCTION MAY ALSO BE CALLED A CORRESPONDENCE (TO EVERY ELEMENT OF S THERE CORRESPONDS A UNIQUE ELEMENT OF T).

This characterization of the relations between spatial response and stimulus configuration will be given in terms of two mathematical functions: (1) a mapping function which defines the correspondence between points of the stimulus and points of the visual space, and (2) a metric which characterizes the internal geometry of the visual space. The constants of this geometry may vary from observer to observer, but repeated and varied experiments strongly indicate that its general character is that of the three-dimensional hyperbolic space of Lobachevski and Bolyai.

A note should be added concerning both the conditions under which the experiments were performed and the method of observation used in viewing the stimulus configurations. All experiments were carried out in a darkroom, thus reducing monocular clues to a minimum. The intensities of the points of light were adjusted to appear equal to the observer but so low that there was no perceptible surrounding illumination. The observer's head was fixed in a headrest and he viewed a static configuration (perception of motion is not considered). The observations were made binocularly and always by allowing the eyes to vary fixation at will over the entire range of the physical configuration until a stable perception of the geometry of the situation was achieved.

Work has been done by other investigators on perceptions arrived at by keeping the eyes in constant fixation on a single point. It is impossible to state *a priori* what relationship, if any, exists between visual space as determined by the "fixed eyes" condition and visual space as determined by using freely roving eyes. However, it seems reasonable to suppose that the fixed eyes condition, owing to the very limited field of distinct vision, would permit only the discovery of local properties of visual space. It is not unlikely that a theory obtained under the fixed eyes condition could be completely subsumed in Luneburg's theory as a theory of the local properties of visual space. On the other hand it is highly probable that the use of the restriction of constant fixation would prevent an understanding of the phenomena associated with the free use of the eyes.*

* IN THIS CONNECTION IT MAY BE WELL TO MENTION THAT CERTAIN OBSERVATIONS CITED IN OGLE[4] AND IN FRY[5,6] ARE MADE WITH THE EYES IN CONSTANT FIXATION. THESE AUTHORS APPARENTLY BELIEVE THAT THEIR RESULTS ARE IN CONTRADICTION TO THE LUNEBURG THEORY. SUCH A CONCLUSION IS NOT WARRANTED BECAUSE OF THE DIFFERENCE IN CONDITIONS. NEITHER AUTHOR HAS CONSIDERED THE POSSIBILITY THAT HIS RESULTS COULD BE CONNECTED TO THE LUNEBURG THEORY THROUGH LOCAL PROPERTIES AND NEITHER HAS ATTEMPTED TO ACCOUNT FOR THE PHENOMENA ASSOCIATED WITH THE FREE USE OF THE EYES.

2. SPECIFICATION OF THE STIMULUS CONFIGURATION - PHYSICAL COORDINATES

In order to give a numerical characterization of a stimulus configuration, its points are located by referring them to a suitable coordinate system,- cartesian, polar or other. The observer's head is assumed to be fixed in normal erect position. The observer's eyes are assumed to be located at points, the rotation centers of the eyes.* A cartesian system is chosen with the origin placed at the point midway between the rotation centers. The y-axis runs laterally through the rotation centers and is oriented positively to the left. The unit of length is fixed by setting the eyes at ± 1 along the y-axis. The x-axis is taken positive in the frontal direction of the median plane. The upward vertical direction is assigned to the z-axis (fig. 1). In this framework we can assign cartesian coordinates (x, y, z) to any point Q in physical space, and so determine its position relative to the observer.

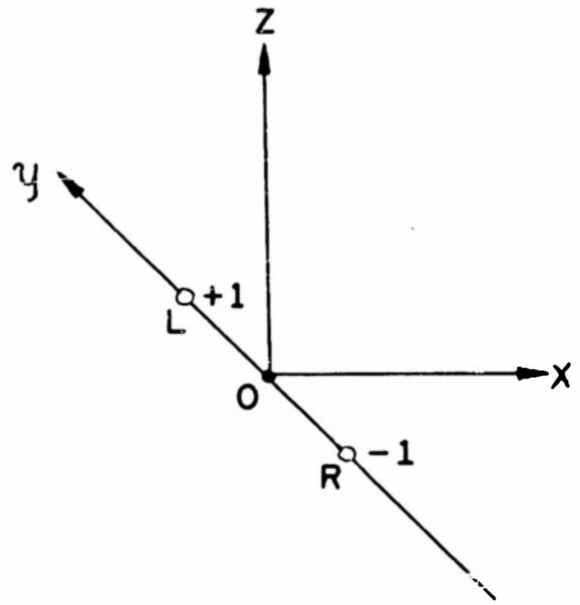


FIG. 1. CARTESIAN COORDINATE SYSTEM FOR PHYSICAL SPACE. L AND R REPRESENT CENTERS OF ROTATION OF LEFT AND RIGHT EYES.

A coordinate system better adapted to our purpose is the bipolar system. Let Q be a physical point anywhere in space and let R denote the right eye and L the left eye (fig. 2). The angle which the plane QLR makes with the horizontal is called the *elevation* θ of the point Q. The angle subtended at Q by the two eyes is called the *bipolar parallax* γ of Q. To completely specify the position of Q we now define a third coordinate, ϕ, the *bipolar latitude*. Let $\bar{x}$ be the axis in the plane of elevation QLR formed by intersection with the median plane (fig. 3). Consider the circle through the three points Q, L, R and let A and B denote the forward and rearward intersections respectively of the elevated axis $\bar{x}$ with the circle. The bipolar latitude is given by $\phi = \sphericalangle ALQ = \sphericalangle ARQ = \sphericalangle ABQ$. The *bipolar coordinates* of Q are the

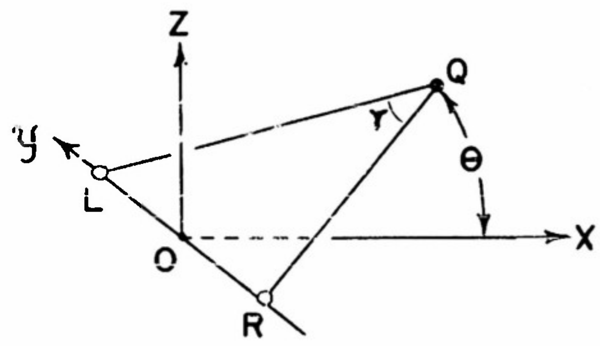


FIG. 2. BIPOLAR PARALLAX AND ELEVATION FOR POINT Q IN THE MEDIAN PLANE.

* THE CONSIDERATIONS BEHIND THIS CHOICE AS OPPOSED TO THAT OF THE NODAL POINTS ARE STATED IN PART II SECTION 1.

above-defined angles:

$$\begin{aligned}
(1) \qquad \gamma &= \measuredangle\, R\,Q\,L && \text{(bipolar parallax)}\\
\phi &= \measuredangle\, A\,B\,Q && \text{(bipolar latitude)}\\
\theta &= \measuredangle\, \bar{x}\,O\,x && \text{(elevation)}
\end{aligned}$$

When the eyes are fixed upon the point Q, the angle γ approximates the angle of convergence of the visual axes and the angle ϕ approximates the average of the inclinations of the two visual axes with respect to the median plane (See Part II, Section 1). For this reason the coordinate γ will often be called the convergence. The angle ϕ will sometimes be called the bipolar azimuth.

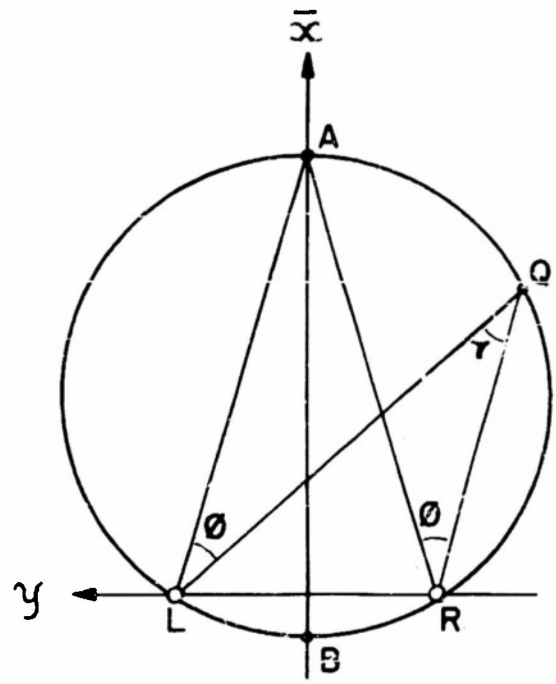


FIG. 3. BIPOLAR PARALLAX AND LATITUDE AS ANGLES IN THE PLANE OF ELEVATION OF POINT Q.

The bipolar coordinates are related to the cartesian coordinates through the transformation equations

$$x = \frac{\cos 2\phi + \cos\gamma}{\sin\gamma}\cos\theta \qquad \tan\gamma = \frac{2\sqrt{x^2+z^2}}{x^2+y^2+z^2-1}$$

$$(2) \qquad y = \frac{\sin 2\phi}{\sin\gamma} \qquad \tan 2\phi = \frac{2y\sqrt{x^2+z^2}}{x^2+z^2-y^2+1}$$

$$z = \frac{\cos 2\phi + \cos\gamma}{\sin\gamma}\sin\theta \qquad \tan\theta = \frac{z}{x}$$

Since most of the investigations have been done in the horizontal plane, we shall generally use only the relations for the horizontal plane

(3)

$$x = \frac{\cos 2\phi + \cos \gamma}{\sin \gamma} \qquad \tan \gamma = \frac{2x}{x^2 + y^2 - 1}$$

$$y = \frac{\sin 2\phi}{\sin \gamma} \qquad \tan 2\phi = \frac{2xy}{x^2 - y^2 + 1}$$

Unless units of length are definitely specified for the cartesian coordinates x,y it is to be understood that the unit of length is half the interpupillary distance of the observer. Similarly, unless it is definitely specified that ϕ and γ are measured in degrees it is to be understood that the angles are given in radian measure.

In many situations it will be useful to employ the approximations

(3a) $$\tan \phi = y/x \qquad \gamma = \frac{2 \cos^2 \phi}{x}$$

which are very good for sufficiently large distances from the observer.

The locus of all points in the horizontal plane which have the same value of γ as Q is the circle passing through Q and the two eyes. This circle is known as the Vieth-Muller Circle through Q and we shall often abbreviate it as VMC. The locus of points ϕ = constant is a hyperbola with the asymptote $\tan \phi = y/x$. In the approximation (3a) we are replacing the hyperbola ϕ = constant by its asymptote and the VMC through Q by a circle passing through Q and tangent to the y - axis at the origin. It is easily seen that the fractional errors made in this approximation are negligible for most purposes. In fact, the estimates

(3b) $$\frac{\Delta \phi}{\phi} < \frac{2}{d^2} \quad \text{and} \quad \frac{\Delta \gamma}{\gamma} < \frac{2}{d^2}$$

where $\Delta \phi$ and $\Delta \gamma$ are the errors and $d \geq 4$ is the distance from the origin, serve very well to show that the approximations are sufficiently accurate for most practical work.

A set of coordinates, which we shall call the *iseikonic coordinates*, particularly useful in analyzing binocular space perception, may easily be defined in terms of the bipolar coordinates. Let γ_o be the least value of γ attributable to any point of the stimulus. If we draw VMC's through all the stimulus points, γ_o will be the value of γ on the outermost VMC. Let ϕ_o and θ_o be values associated with suitable directions of reference. To a point having the bipolar coordinates (γ, ϕ, θ) we associate iseikonic coordinates,

$$\Gamma = \gamma - \gamma_0$$

(4) $$\Phi = \phi - \phi_0$$

$$\Theta = \theta - \theta_0$$

These coordinates will generally have to be specified anew for each change of stimulus.

3. THE METRIC NATURE OF VISUAL SPACE

The mathematical characterization of the visual space is founded upon a set of observations in conjunction with a limited number of mathematical assumptions of considerable heuristic appeal. From these fundamentals it is possible to achieve by deductive process Luneburg's simple characterization of the geometry of visual space. In fact, he presented strong evidence that the visual space is a metric space, finitely compact, convex and homogeneous. Our further work supports this conclusion.

3a. Visual Orientation

One of the curious facts of binocular perception is that the observer is not ordinarily aware of any bipolarity. Sensed distances from the observer are treated as though viewed from a point center of reference. This situation is described by placing the origin of visual coordinates at this "egocenter" of the observer.

The observer is, however, aware of the orientations lateral, vertical and frontal. In the visual space we may then take three subjective planes of orientation perpendicular to these axes - the sensed median, horizontal and frontal planes through the origin. The axes in the visual space are the intersections of the three principal subjective planes. Let (ξ, η, ζ) be coordinates chosen to represent these subjective orientations. The origin $\xi = \eta = \zeta = 0$ represents the subjective center of observation. The ξ - axis is positive in the frontal direction; the η - axis, in the direction left; the ζ - axis, in the direction vertically upwards. The subjective horizontal, median and frontal planes are given by the respective equations, $\zeta=0$, $\eta=0$, $\xi=0$.

The positional orientation of the observer is generally such that he brings the subjective planes into the proper orientation with respect to objective physical space.

This coordination between the visual and proprioceptive senses is not absolute, however. It may easily be disarranged in an airplane or sea-going vessel. We shall see, in fact, that the assumption of the customary correspondence between objective and subjective orientations is not necessary for our theory.

3b. Perception of Distance

A configuration consisting of isolated points Q_1, Q_2, Q_3, ... is sensed as a distribution of points P_1, P_2, P_3, ... in a three-dimensional continuum. An observer obtains rather definite impressions of the distance of the points from one another and from the observational center. The sizes of these sensed distances may readily be compared. Thus if (P_i, P_j) denotes the sensed distance between any two points P_i and P_j, we find for any two pairs of points P_1, P_2 and P_3, P_4 that relations of inequality such as

$$(P_1, P_2) > (P_3, P_4) \quad \text{or} \quad (P_1, P_2) < (P_3, P_4)$$

are easily perceived. The sensed relations of equality and inequality are quite stable for a given observer. In other words, the inequality signs are determined to a high degree of correlation by the physical coordinates of the stimulating points Q_1, Q_2, Q_3, Q_4.

3c. Perception of Straightness

A sense of alignment is one of the strong characteristics of visual perception. We quickly perceive whether or not three points lie on a straight line. Furthermore, physical points can be arranged so as to result in the perception of a straight line for every orientation and position in the visual space. Given an arbitrary pair of points, it is possible to arrange others along a curve which will be perceived as the extended straight line joining the points. Perhaps it would be well to emphasize that the perception of straightness may arise from physical curves* which are not physically straight but actually have marked curvature. (See Section 5e)

3d. The Psychometric Distance Function

The observations 3b and 3c are a strong indication that the visual space is a mathematical metric space. This means that we can assign positive numerical values

*THE WORD CURVE AS USED HERE IS TAKEN IN THE TECHNICAL MATHEMATICAL SENSE. A CURVE IS A ONE-DIMENSIONAL CONTINUOUS MANIFOLD. THUS A STRAIGHT LINE IS A KIND OF CURVE. IT HAS ZERO CURVATURE EVERYWHERE.

to sensed distances so that the numbers satisfy inequalities in agreement with the perceived relations of sensed distances. Such a coordination of a number $D(P_1, P_2)$ to the sensed distance between a pair of points P_1, P_2 is called a distance function or metric if it satisfies the following conditions:

(a) $D(P, P) = 0$. A perceived point has zero distance from itself.

(b) $D(P_1, P_2) = D(P_2, P_1) > 0$, if $P_1 \neq P_2$. To each perceived pair of distinct points there is assigned a positive value of distance independent of the order in which the points are considered.

(c) $D(P_1, P_2) + D(P_2, P_3) \geq D(P_1, P_3)$ for any three points P_1, P_2, P_3. We shall say in particular that three points are on a *straight line* if and only if the equality relationship holds.

When we say that the function $D(P_1, P_2)$ corresponds to sensed distance we mean that it must satisfy the further conditions:

(d) If P_1, P_2 and P_3, P_4 are any two pairs of perceived points, then

$$D(P_1, P_2) \gtreqless D(P_3, P_4)$$

according to whether the sensed distances are correspondingly related,

$$(P_1, P_2) \gtreqless P_3, P_4).$$

(e) If P_1, P_2, P_3 are perceived as being arranged in that order on a straight line, then

$$D(P_1, P_2) + D(P_2, P_3) = D(P_1, P_3),$$

and conversely.

A function satisfying conditions (a) to (e) is called a *psychometric distance function* or simply a *metric* for visual space. Our problem can be reduced to the determination of such a function in the terms of the physical coordinates of the stimulating points. Quite clearly, the physical distance relations among the stimulating points will not describe a metric for visual space. Although physical distance satisfies (a) to (c) it can not satisfy (d) or (e) since, for one thing, the physically straight lines are not generally the same as the visually straight lines. To keep these distinctions clear, the curves in physical space which are perceived as straight will be called *visual geodesics* or simply *geodesics*.

The function $D(P_1, P_2)$ is not completely determinate, for if $D(P_1, P_2)$ satisfies conditions (a) to (e) so does the function $C \cdot D(P_1, P_2)$ where C is any positive constant whatever. Yet, under certain general mathematical assumptions, this can

be proved to be the only indeterminacy possible. These assumptions are:

(f) The visual space is *finitely compact.*

Every bounded infinite sequence of points has a limit point; i.e. for every infinite sequence of points P_ν ($\nu = 1, 2, 3, \ldots$) satisfying the condition $D(P_o, P_\nu) < M$ for some point P_o and positive constant M, there exists a subsequence P_{ν_k} ($k = 1, 2, 3 \ldots$) and a point P of the visual space such that $D(P, P_{\nu_k}) \rightarrow 0$.

(g) The visual space is *convex.*

Between every pair of points P_1, P_2 ($P_1 \neq P_2$), there is a point P_3 on the straight segment joining P_1 to P_2; i.e., there exists a point P satisfying

$$D(P_1, P_3) + D(P_3, P_2) = D(P_1, P_2).$$

The proof that, under these assumptions, the metric is completely determinate to within a constant factor is given in Luneburg[7].

Although the assumptions (f), (g) can not be verified by experiment since the proof would require infinitely many tests, they do coincide with our customary convictions about visual perception.

Since a distance function may be determined exactly to within a constant factor, it follows for a given stimulus, that the proportions of distance are unique. In other words, the ratio $D(P_1, P_2) / D(P_3, P_4)$ of two sensed distances is a uniquely determined function of the four stimulating points in question and does not depend upon the particular distance function we use. In this way the metric establishes a fixed relationship between the objective physical stimulus and the subjective perception. This relation is a function of no other variables than the coordinates of the stimulus. Any parameters in this relationship which are not physical coordinates must be constant factors of the observer, characterizing his visual reactions to external stimuli.

3e. The Homogeneity of Visual Space*

The visual space has two properties which are familiar from common experience but have not been treated experimentally. For this reason these properties are stated here as hypotheses. The first of these properties is:

(h) The visual space is locally euclidean.

*FOR A FAIRLY COMPLETE ACCOUNT OF THE MATHEMATICAL KNOWLEDGE IN THIS SECTION, SEE BUSEMANN[8]

In other words, the euclidean laws hold to any desired degree of approximation in sufficiently small regions of space. The earth, considered as a spherical surface, is a familiar example of a space having this kind of property. In surveying a small area it suffices to use the euclidean laws of ordinary trigonometry, but for navigating over great distances only spherical trigonometry will do. The locally euclidean property of visual space explains why we may notice no distortion in viewing small geometrical diagrams frontally. The property (h), together with the properties of finite compactness and convexity, forms a necessary and sufficient condition that the space be *riemannian*.

The second property which we postulate is that sensorially plane surfaces exist in any given orientation and localization. The visual perception of planeness is such that the visual geodesic connecting any two points of a sensory plane does not anywhere depart from that plane. Any three physical stimulus points can be imbedded in one surface, and only one, which gives the impression of planeness. All the statements concerning the nature of the visual planes can be summarized in one:

(i) The visual space is a desarguesian geometry.

From the propositions (a) to (i) it can be proved that the visual space is homogeneous. *The binocular visual space is one of the riemannian spaces of constant gaussian curvature.*

A mathematical consequence of the homogeneity of visual space is that the metric must be one of three simple kinds. For the (ξ, η, ζ) coordinate system used by Luneburg, the psychometric distance function $D = D(P_1, P_2)$ is given by the formula:

(5)
$$\frac{2}{(-K)^{1/2}} \sinh\left[\frac{(-K)^{1/2}}{2}\ \frac{D}{C}\right] = \left[\frac{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\zeta_1 - \zeta_2)^2}{(1 + \frac{K}{4}\rho_1^{\,2})\ (1 + \frac{K}{4}\rho_2^{\,2})}\right]^{1/2} = F(P_1, P_2)$$

where (ξ_1, η_1, ζ_1) and (ξ_2, η_2, ζ_2) are the coordinates of P_1 and P_2 respectively and where $\rho_i^{\,3} = \xi_i^{\,2} + \eta_i^{\,2} + \zeta_i^{\,2}$ $(i = 1, 2)$. The constant K may be interpreted as the generalized gaussian curvature of the space. The constant C is the arbitrary constant factor of indeterminacy in the metric. If K is allowed to approach zero from either side, the formula (5) becomes

(5a)
$$\frac{D}{C} = F\ (P_1, P_2) \qquad (K = 0)$$

and the relation obtained is simply the familiar euclidean metric. If K is positive, the formula (5) is usually written more conventionally as

$$\frac{2}{K^{1/2}} \sin\left[\frac{K^{1/2}}{2}\frac{D}{C}\right] = F(P_1, P_2) \qquad (K > 0) \tag{5b}$$

The metric (5b) is that of elliptic geometry. The two-dimensional case is familiar to us as the geometry on the surface of a sphere.

Negative K gives us the hyperbolic geometry of Lobachevski and Bolyai. The evidence of our experimental studies indicates repeatedly and in a variety of ways that the geometry of visual space is, in fact, just this hyperbolic geometry.

If we interpret (ξ, η, ζ) as cartesian coordinates we can map visual space in a euclidean space. Visual distances could not be represented correctly by distances on the map unless the metric were euclidean, since the three metrics are clearly not proportional. In fact, we know that in making a map of the earth (elliptic case) on a euclidean sheet of paper we cannot avoid distorting distances. The map described by the (ξ, η, ζ) coordinates does, however, have one clear advantage,-it is conformal. This means that perceived angles will be exactly represented by angles on the map. As a matter of convenience in formulation we prefer to use an equivalent set of coordinates, polar coordinates (r, φ, ϑ) in visual space.

With Luneburg, we set

$$\begin{aligned} \xi &= \rho \cos\varphi \cos\vartheta \\ \eta &= \rho \sin\varphi \\ \zeta &= \rho \cos\varphi \sin\vartheta \end{aligned} \tag{6}$$

However, we replace ρ in the hyperbolic case by

$$\rho = \frac{2}{(-K)^{1/2}} \tanh\frac{r}{2} \qquad (K < 0) \tag{7}$$

For the euclidean and elliptic cases we set

$$\rho = r \qquad (K = 0) \tag{7a}$$

$$\rho = \frac{2}{K^{1/2}} \tan\frac{r}{2} \qquad (K > 0) \tag{7b}$$

The radial coordinate r is to be interpreted as a quantity measuring sensed distance from the observer. It is never to be taken as an absolute of sensation, but only as a correct description of relative distance when taken together with other values. In any case, all points perceived as having the same distance from the

observer must be assigned the same value of r.

The equation r = constant represents a sphere about the egocenter. The coordinate ϑ simply represents the perceived angle of elevation from the subjective horizontal. Thus on the sphere r = constant, the curves ϑ = constant represent meridans of longitude passing through poles on the left and right of the egocenter. In the same way, the curves φ = constant represent parallels of latitude on the visual sphere. The visual sphere r = constant can be conceived in this way as the earth with its axis oriented horizontally. By employing the coordinate transformations (6), (7), we obtain the hyperbolic metric in terms of the visual polar coordinates in the form

$$\cosh\frac{D}{C} = \cosh r_1 \cosh r_2 - \sinh r_1 \sinh r_2 \, f(\varphi_1,\varphi_2;\vartheta_1,\vartheta_2), \quad (K < 0) \tag{8}$$

where

$$f(\varphi_1,\varphi_2;\vartheta_1,\vartheta_2) = \cos(\varphi_2-\varphi_1) - \cos\varphi_1\cos\varphi_2\,[1-\cos(\vartheta_2-\vartheta_1)]$$

for the euclidean and elliptic cases we have

$$\left(\frac{D}{C}\right)^2 = r_1^{\,2} + r_2^{\,2} - 2r_1r_2\, f(\varphi_1,\varphi_2;\vartheta_1,\vartheta_2), \quad (K = 0) \tag{8a}$$

and

$$\cos\frac{D}{C} = \cos r_1 \cos r_2 + \sin r_1 \sin r_2\, f(\varphi_1,\varphi_2;\vartheta_1,\vartheta_2), \quad (K > 0) \tag{8b}$$

It will be seen that equations (8) and (8b) may be transformed into each other by replacing the sensed radial distance r with its imaginary counterpart ir. Two-dimensional hyperbolic space might in this way be interpreted as the geometry on the surface of a sphere of "imaginary radius".*

It is well-known that there is an absolute measure of length in elliptic geometry. In the two-dimensional case, for example, it is possible to represent the elliptic geometry isometrically on the surface of a sphere. The radius of the mapping sphere may then be used as an absolute measure of length. If the radius of curvature in this representation is taken as unity, then we must take C = 1 in (8b). By the analogy cited above, it is possible to specify an absolute measure in the hyperbolic geometry, too. Gauss remarked that he wished the physical world were not euclidean for then there would be *a priori* an absolute measure of length.** We shall, by analogy with the elliptic case, take C = 1 in (8). However, it should be remembered that this particular metric for visual space is only one choice out of a possible one-parameter infinity.

* THIS IS NOT TO BE CONSTRUED AS HAVING ANY SIGNIFICANCE DEEPER THAN THAT IMPLIED BY THE SUBSTITUTION OF ir FOR r IN (8b).

**GAUSS, LETTER TO F.A.TAURINUS (1824): "ICH HABE DAHER WOHL ZUWEILEN IN SCHERZ DEN WUNSCH GEAUSSERT, DAS DIE EUKLIDISCHE GEOMETRIE NICHT DIE WAHRE WÄRE, WEIL WIR DANN EIN ABSOLUTES MAASS A PRIORI HABEN WURDEN."
(SEE ENGEL F. AND STÄCKEL, P., THEORIE DER PARALLELLINIEN. LEIPZIG, 1895, FOR ENTIRE LETTER).

3f. Plane Trigonometry of the Visual Space*

If we let $\vartheta = 0$ in the formulas (8) and consider the metric relations between the sides and angles of triangles, we shall compile a set of useful relations which may be used to measure the visual space just as we use trigonometry to measure the physical world. Let the scale factor C in (8) be unity. Denote by a, b, c the perceived lengths of the sides of a triangle and let A, B, C denote the perceived sizes of the opposite vertex angles. By employing the metric (8) it is possible to derive the analog to the law of cosines for the hyperbolic case:

(9) $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C \qquad (K < 0)$

The corresponding rules for the euclidean and elliptic cases are

(9a) $c^2 = a^2 + b^2 - 2ab \cos C \qquad (K = 0)$

and

(9b) $\cos c = \cos a \cos b + \sin a \sin b \cos C \qquad (K > 0).$

The "Pythagorean theorem" for hyperbolic right triangles is obtained by setting $C = 90^\circ$ in (9):

(10) $\cosh c = \cosh a \cosh b \qquad (K < 0)$

and in the two other cases we have

(10a) $c^2 = a^2 + b^2 \qquad (K = 0)$

(10b) $\cos c = \cos a \cos b \qquad (K > 0)$

In fact, we may set down the usual laws for the angle functions of right triangles in all three geometries:

	K < 0	K = 0	K > 0
(11)	$\cos A = \dfrac{\tanh b}{\tanh c}$	$\dfrac{b}{c}$	$\dfrac{\tan b}{\tan c}$
(12)	$\sin A = \dfrac{\sinh a}{\sinh c}$	$\dfrac{a}{c}$	$\dfrac{\sin a}{\sin c}$
(13)	$\tan A = \dfrac{\tanh a}{\sinh b}$	$\dfrac{a}{b}$	$\dfrac{\tan a}{\sin b}$
(14)	$\dfrac{\cos A}{\sin B} = \cosh a$	1	$\cos a$
(15)	$\cot A \cot B = \cosh c$	1	$\cos c$

For small triangles it is easy to see that the hyperbolic and elliptic rules both approach the euclidean one.

* THE READER IS REFERRED TO COXETER[9] AND CARSLAW[10]

The law of sines in hyperbolic trigonometry is especially simple:

$$\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \frac{\sinh c}{\sin C} \qquad (K < 0) \tag{16}$$

For the other geometrics we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad (K = 0) \tag{16a}$$

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \qquad (K > 0) \tag{16b}$$

4. RELATION OF VISUAL TO PHYSICAL SPACE

At the Dartmouth Eye Institute, Ames succeeded empirically in constructing a sequence of distorted rooms which could hardly be distinguished from a given rectangular room with respect to binocular vision. At first Luneburg[1] suggested that the construction of these rooms could be mathematically derived from the rectangular original by employing a certain kind of transformation which he called an *iseikonic* transformation (Fig. 4). This transformation was determined by the assumption that the rotatory motion of the eyes in looking from point to point of a configuration was the sole determining factor in the perception of the relative positions of the points.* Subsequently, he discarded this notion in favor of the idea that the fixation angles themselves, rather than only the changes in fixation angles, were significant in binocular perceptions. The distorted rooms could then be accounted for by translatory displacements in the hyperbolic visual space. In each case he obtained a one-parameter family of distorted rooms which would account for the characteristic shape of the Ames constructions (See Luneburg[11]).

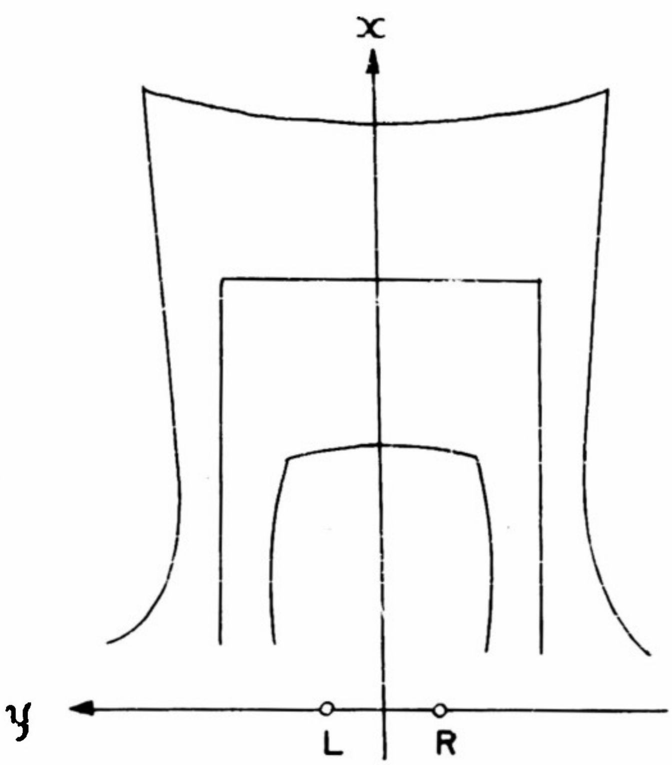


FIG. 4. BINOCULARLY INDISTINGUISHABLE CONFIGURATIONS.

*WHETHER IT IS THE SEQUENCE OF RETINAL IMAGES, OR THE MUSCULAR ACTION OR BOTH TOGETHER WHICH INFORM US IN THIS WAY, IS IRRELEVANT HERE.

The two hypotheses do give measurable differences and it would be possible to discover by experiment which is correct. However, experimental evidence obtained in other ways has led us to utilize the earlier point of view.

4a. The Iseikonic Transformations

According to Luneburg's earlier hypothesis, if the bipolar coordinates of all points in a given stimulus were changed by constant amounts λ, μ, ν by means of the transformation

$$\gamma' = \gamma + \lambda$$
$$\phi' = \phi + \mu \tag{17}$$
$$\theta' = \theta + \nu$$

then, to any one observer, the new stimulating configuration would yield the same perceptions as the original configuration. In particular the Ames rooms could be constructed by employing the special transformations

$$\gamma' = \gamma + \lambda$$
$$\phi' = \phi \tag{17a}$$
$$\theta' = \theta$$

One reason Luneburg gave for discarding this hypothesis was the fact that two segments having the same disparities $\Delta\gamma$, $\Delta\phi$, $\Delta\theta$ between their endpoints are not necessarily perceived as having equal lengths. However, this misses the fact that in this case the two segments are being compared with each other in the same stimulus configuration. It is when we transform the entire stimulus into another and the entire stimulus presented is either the original or the transformed one but not both together, that we may say the perceptions arising from the new are the same as those arising from the old.

Aside from the evidence of the Ames constructions, we shall be able to give quantitative verification of the relation of the iseikonic transformations to perception. The data are given in Part II Section 2 in the studies on the Oblique Geodesics, the Double Vieth-Müller Circles and the Equipartitioned Parallel Alleys.

The modification of our ideas presented here consists entirely of introducing into the theory Luneburg's earlier conception of the role of the iseikonic transformations in visual space perception. What has been changed is simply the idea of the way physical space is mapped into visual space. Luneburg's conception of the internal struc-

ture of visual space is left unaltered, and in fact we use the same sort of method in measuring within the visual space. Three hypotheses H_1, H_2, H_3 are added subsequently in connection with the change in the mapping but, in any case, these are collateral hypotheses, not at all essential to the main argument of the theory.

4b. The Vieth-Müller Torus-Perceived Radial Distance

It will be recalled that a Vieth-Müller Circle (VMC) is one of the circles γ = constant in the horizontal plane and is a circle passing through the eyes. The Vieth-Müller Torus is the three-dimensional surface obtained by rotating this circle about the axis through the eyes. It looks a bit like an apple with the eyes at the bottom of the indentations at either end (Figure 5). Luneburg observed that a set of points arranged in the horizontal plane so as to give the perception of a circle of points at the same fixed distance from the observer, approximates fairly well an arc of a Vieth-Müller circle. Subsequent experiments have shown that this observation is substantially true. Consistent deviations seem to exist, but there is insufficient statistical evidence to warrant replacing γ by a more complicated coordinate.

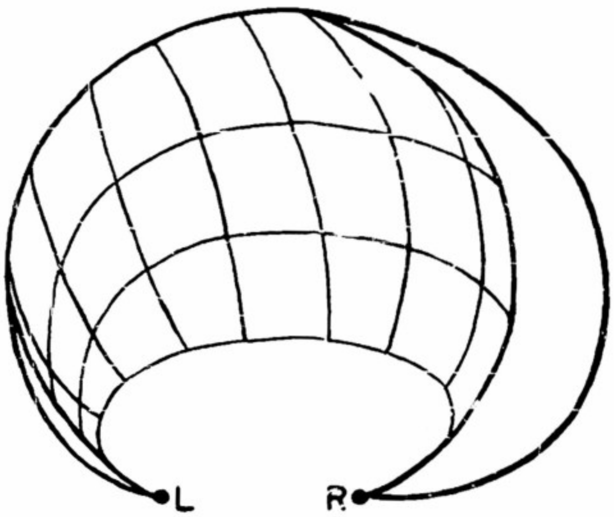


FIG. 5. SEGMENT OF A VIETH-MÜLLER TORUS

On the basis of this evidence Luneburg expressed the hypothesis,

H_1 A Vieth-Müller Torus is perceived as a sphere with the observer at its center.

In mathematical language, the hypothesis asserts that the toruses γ = constant in physical space are mapped as spheres in the visual space. It is possible that this hypothesis may have to be modified. For example, the well-known observation that the zenith of the night sky appears to be closer than the horizon (although such an observation may not be absolutely free from intellectual clues) indicates that the hypothesis is worth re-examining.*

*cf. LUNEBURG[3] p. 633.

Since most of the work for this study has been done in the horizontal plane we shall leave this point as subject to investigation by further experiment.

A convenient hypothesis for what follows, although not absolutely essential to the theory, is

H_2 Among all the points of a given stimulus configuration those which have the same γ are perceived as being equidistant from the observer.

It may seem superfluous to make this further assumption of the role of the VMC. Nonetheless, the idea that the perception of equal depth will not be affected by adding other stimulus points at random distances is not an *a priori* certainty.

From the hypothesis H_2 we see that the perceived radial distances for a given stimulus and its iseikonic equivalents depend only on the differences in γ among the points of the stimulus. In particular, if γ_0 is the value associated with the greatest perceived distance in the stimulus, the value of radial distance for any other point with coordinate γ will depend only on $\gamma - \gamma_0$. For stimuli which are not connected by iseikonic transformation we state the "hypothesis of the limiting sphere",

H_3 The perceived ratio of radial distance for any point of a stimulus to that of the point of perceived greatest radial distance depends only on the difference in convergence between the two points, *independently of the stimulus.*

The name of the hypothesis stems from the fact that it is equivalent to the assertion that the farthermost point in every stimulus is mapped onto a limiting sphere $r = \omega$ in the visual space, where ω is a personal constant independent of the stimulus.

The hypothesis H_3 is in accord with the fact that in all observations and experience the visual space appears to be finite. There is nothing in our perceptions corresponding to the ideas of "infinitely far away" or "infinitely large". This hypothesis is given some support by certain experimental observations of equipartitioned alleys, and by the fact that the computed values of ω for different kinds of experiment are in approximate agreement (Part II, Sec. 3).

4c. Perceived Direction

Two points Q_1 and Q_2 will lead to the perception of two points P_1, P_2 in the same direction at different distances only if their angular coordinates θ and ϕ are the same. Thus the hyperbolas in physical space, determined by the equations θ = constant, ϕ = constant, are mapped into radial lines, ϑ = constant, φ = constant, of visual space. Equal changes in ϕ and θ are perceived as equal changes in φ and ϑ. Since, the physical and visual orientations of the principal planes will generally be in agreement, we may, when this orientation is preserved, set $\varphi = \phi$, $\vartheta = \theta$. However, it is sufficient for our purposes to state that perceived differences in φ and ϑ in looking from point to point of a configuration are equal to the physical differences in ϕ and θ.

4d. The Sensory Role of the Iseikonic Coordinates

From the preceding remarks it is quite plain that the iseikonic coordinates

$$\begin{aligned} \Gamma &= \gamma - \gamma_o \\ \Phi &= \phi - \phi_o \\ \Theta &= \theta - \theta_o \end{aligned} \tag{18}$$

are highly suitable for the description of perceptual phenomena in the visual space. In the first place they are invariant under iseikonic transformation as are the perceived metric relationships among the points of a configuration. If γ_o is the coordinate of the farthest sensed point and ϕ_o and θ_o are suitable directions of reference, we may set

$$\begin{aligned} r &= r(\Gamma) \\ \varphi &= \Phi \\ \vartheta &= \Theta \end{aligned} \tag{19}$$

The function r (Γ) is a constant characteristic of the observer. In particular, so is the special value

$$\omega = r\,(0) \tag{19a}$$

Under the assumptions of the foregoing analysis we have reduced the problem of determining the coordination between visual and physical space to the determination of the

single function $r = r(\Gamma)$. The function $r(\Gamma)$ is a personal characteristic (i.e., a constant such as Luneburg predicated) of the observer. If our assumptions are correct, a complete description of the observer's binocular visual space can be supplied once the function $r(\Gamma)$ is determined.

5. EXPERIMENTAL METHODS FOR DETERMINING $r(\Gamma)$ AND RELATED EXPERIMENTS

The rules of trigonometry given in Section 3f may be used to measure the visual space. In Part II we shall discuss several relevant experiments which have been performed in this laboratory, together with a detailed account of the technics, apparatus and results obtained. In the present section only a general description of various experiments related to the theory will be presented.

Convenience has led us to restrict our work to the use of stimulus configurations in the horizontal plane, $\Theta = 0$. Although it would be desirable to complete the evidence by performing experiments in all three dimensions, there is some foundation, in theory, for the hope that conclusions based on results obtained in the horizontal plane may have validity also for the three-dimensional case.

As a matter of consistent notation, points of the stimulus configuration will be denoted by the letters $Q_1, Q_2, Q_3, \ldots$ and the corresponding perceived points by the letters $P_1, P_2, P_3 \ldots$

5a. Parallel and Distance Alleys

The most striking evidence that visual space is non-euclidean lies in the distinction in visual perception between apparently parallel straight lines and curves of apparent equidistance. This difference was first reported by Blumenfeld[12]. The experiment is quite simple. Two lights are fixed at the points $Q_1^+ = (\gamma_1, \phi_1)$ and $Q_1^- = (\gamma_1, -\phi_1)$, equidistant from the observer and symmetric to the median. Other lights are then introduced successively in pairs $Q_n^{\pm}$ at predesignated stations approaching the observer. The observer is asked to adjust the pair $Q_n^{\pm}$ according to two different sets of instructions:*

*For this experiment, also for those discussed in sections 5b and 5c, the complete instruction is given in Part II.

"(a) Adjust the lights $Q_2^{\pm}$, $Q_3^{\pm}$, . . . , $Q_n^{\pm}$ until the two rows of lights appear to be straight, parallel to each other and parallel to the median.

"(b) With only the fixed lights $Q_1^{\pm}$ left on, set the pair $Q_n^{\pm}$ to appear symmetric to the median and to have the same apparent separation as the two fixed lights".

The result of experiment (a) is called a *parallel alley:* of experiment (b), a *distance alley*. If the geometry were euclidean, the two instructions should lead to the same result. We should obtain only one pair of curves. These curves would be symmetric to the median, would have the appearance of being straight, parallel to each other and the median, and would be equidistant throughout their lengths. This is not, in fact, the case. For all observers who appear to understand the instructions* the curves (a) and (b) are quite different from each other. If the curves (b) are illuminated for the observer after their pairwise construction, they appear to be neither parallel nor straight. For these observers the parallel alleys fall nearer to the median than the distance alleys. (Fig. 6).

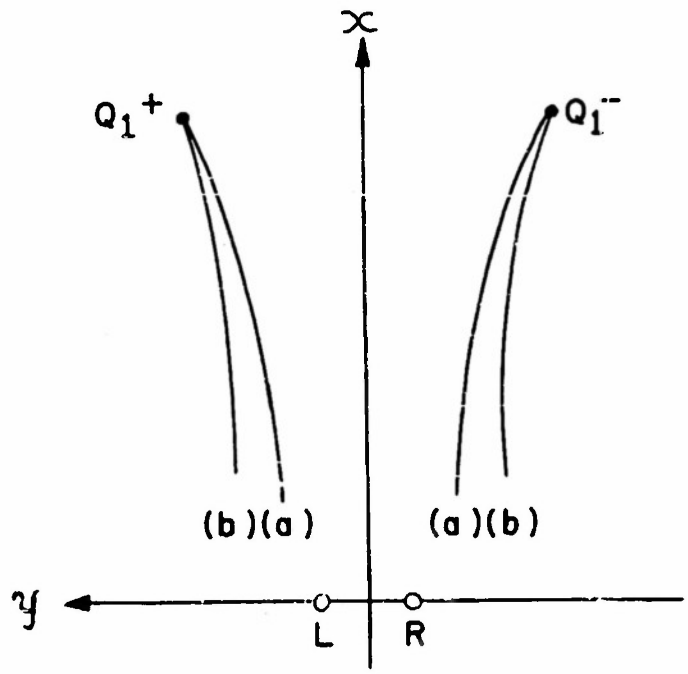


FIG. 6. BLUMENFELD ALLEYS: (a) PARALLEL ALLEY; (b) DISTANCE ALLEY.

For iseikonic coordinates in these experiments we take $\Gamma = \gamma - \gamma_1$ and $\Phi = \phi$.

In the visual space, Luneburg characterizes the parallel alleys as the visual geodesics which are sensed as being perpendicular to the subjective frontal plane (Fig. 7). The equation for sensed straight lines satisfying this requirement is simply obtained. Let $P_1^{\pm} = (r_1, \pm \varphi_1)$. We have $r_1 = \omega$, $\varphi_1 = \phi_1$. Let $P = (r, \varphi)$ be a variable point on the alley through P_1^+ and let Y denote the radial distance of the intercept of the alley with the η-axis. From the right-triangle formula (11) we obtain

*FOR AN APPRECIATION OF THE DIFFICULTY HERE, SEE HARDY, RAND, RITTLER[13]

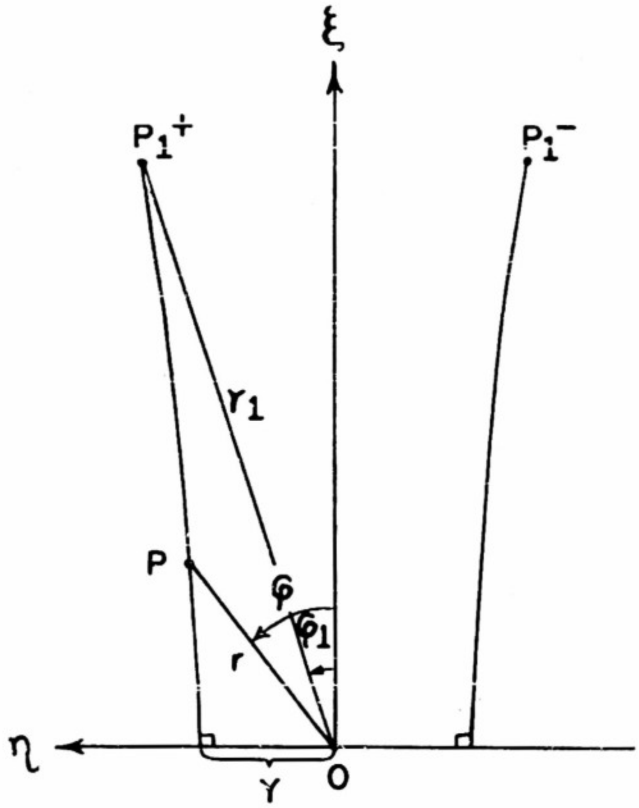


FIG. 7. REPRESENTATION OF A PARALLEL ALLEY IN VISUAL COORDINATES.

in the hyperbolic case

$$\cos\left(\frac{\pi}{2} - \varphi\right) = \sin \varphi = \frac{\tanh Y}{\tanh r}$$

The constant tanh Y is related to the coordinates of the fixed point by $\tanh Y = \sin \varphi_1 \tanh \omega$. The equation of the parallel alley in hyperbolic geometry is therefore

(20) $\tanh r \sin \varphi = \tanh \omega \sin \varphi_1$ $(K < 0)$

For the other two geometries the same method gives

(20a) $r \sin \varphi = \omega \sin \varphi_1$ $(K = 0)$

(20b) $\tan r \sin \varphi = \tan \omega \sin \varphi_1$. $(K > 0)$

The distance alleys, on the other hand, may be characterized as the loci of constant perceived distance d from the median (Fig. 8.) For a variable point

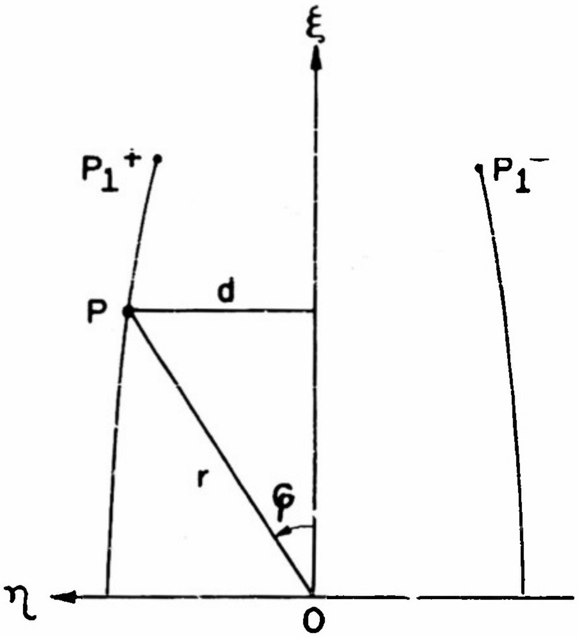


FIG. 8. REPRESENTATION OF A DISTANCE ALLEY IN VISUAL COORDINATES.

$P = (r, \varphi)$ on the left-hand alley, we obtain from (12) $\sin\varphi = \sinh d / \sinh r$ in the hyperbolic case. From the condition that the alley go through $P = (\omega, \varphi_1)$ we find

(21) $$\sinh r \sin\varphi = \sinh\omega \sin\varphi_1. \qquad (K < 0).$$

For the other two cases the equations for the distance alleys are

(21a) $$r \sin\varphi = \omega \sin\varphi_1 \qquad (K = 0)$$

(21b) $$\sin r \sin\varphi = \sin\omega \sin\varphi_1. \qquad (K > 0)$$

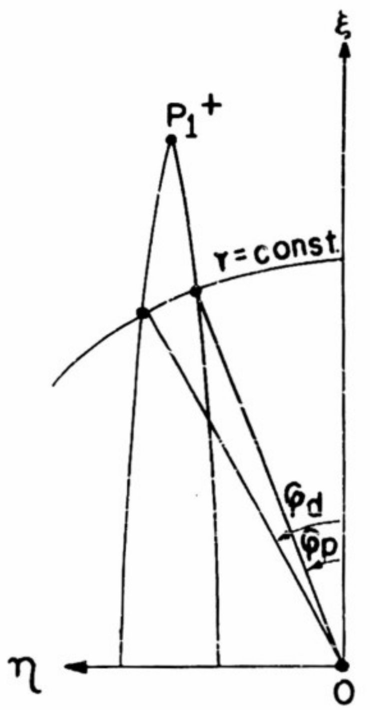


FIG. 9. REPRESENTATION SHOWING $\varnothing_p$ AND $\varnothing_d$ FOR A GIVEN VALUE OF r.

In the euclidean case, as we know, the parallel and distance alleys are the same and this geometry does not account for the experimental observation. Now, if we let φ_p be the angular coordinate on the parallel alley and φ_d be that on the distance alley for a given value of $r = \omega$ (Fig. 9), we find from (20) and (21),

(K<0)

$$\sin\varphi_1 = \frac{\tanh r \sin\varphi_p}{\tanh\omega} = \frac{\sinh r \sin\varphi_d}{\sinh\omega}$$

and (K>0)

$$\sin\varphi_1 = \frac{\tan r \sin\varphi_p}{\tan\omega} = \frac{\sin r \sin\varphi_d}{\sin\omega}$$

Now, since $r \leq \omega$, the above equation for the hyperbolic case yields

$$\text{(22)} \qquad \frac{\sin \varphi_p}{\sin \varphi_d} = \frac{\cosh r}{\cosh \omega} < 1 \qquad (K < 0)$$

This implies that $\varphi_p < \varphi_d$ and the parallel alley must be inside the distance alley. For the elliptic case on the other hand we find

$$\frac{\sin \varphi_p}{\sin \varphi_d} = \frac{\cos r}{\cos \omega} > 1 \qquad (K > 0)$$

Consequently, $\varphi_p > \varphi_d$ and the parallel alley lies outside the distance alley.

Clearly, *the hyperbolic case is the only one that can fit this experimental evidence.* We shall find that other experimental tests of the question lead to the same conclusion. For this reason we shall no longer follow this parallel presentation of the three cases, and we shall employ only the hyperbolic geometry. The reader will find it not difficult to carry out the analogous reasoning for the other cases if he wishes to do so.

The alley experiments may be used not only as a means of indicating the hyperbolic character of the geometry, but also to calculate the function $r(\Gamma)$. Consider the VMC corresponding to a given value of Γ (y_0 is already specified) and let r be the perceived radial distance corresponding to Γ. The point (r, φ_d) on the distance alleys satisfies equation (21) and, hence,

$$\sinh^2 r = \sinh^2 \omega \, \frac{\sin^2 \varphi_1}{\sin^2 \varphi_d}.$$

The coordinates (r, φ_d) are related to the coordinates of the point (r, φ_p) on the parallel alley by means of the equation (22) which yields the relation

$$\cosh^2 r = 1 + \sinh^2 r = \cosh^2 \omega \, \frac{\sin^2 \varphi_p}{\sin^2 \varphi_d}.$$

By eliminating $\sinh^2 r$ from the two equations and setting $\sinh^2\omega = \cosh^2\omega - 1$ we obtain an equation for ω:

$$\text{(23)} \qquad \cosh^2 \omega = \frac{\sin^2\varphi_d - \sin^2\varphi_1}{\sin^2\varphi_p - \sin^2\varphi_1} = \frac{\cos 2\varphi_1 - \cos 2\varphi_d}{\cos 2\varphi_1 - \cos 2\varphi_p}$$

Having determined the value of ω from equation (23), the values of r for other values of Γ may be determined from equation (20) or (21) by taking points on the respective alleys.

5b. The Double Vieth-Müller Circles

Experiments utilizing points of light set on two Vieth-Müller circles of different bipolar parallax were described by Luneburg[3].

(i) *The Three-Point Experiment.* This experiment has given most uniform results in favor of the hypotheses that the visual space is hyperbolic. Luneburg has shown also that the experimental results lend further support to the hypothesis of constant curvature.

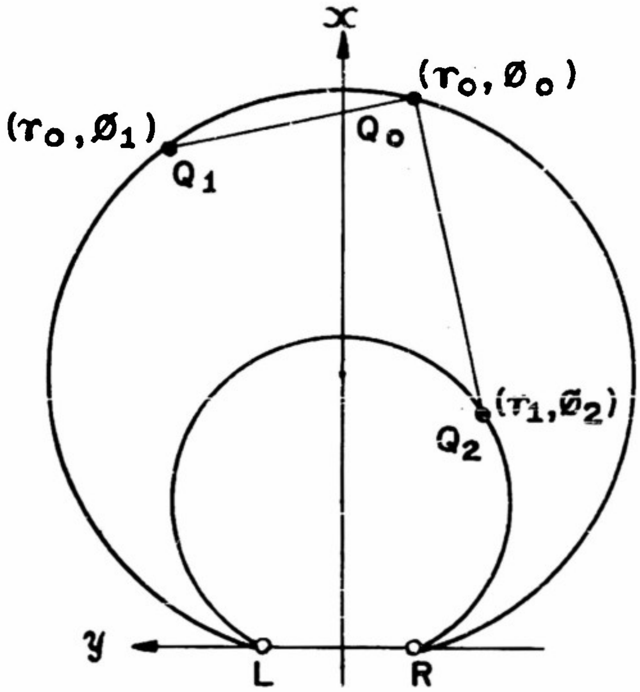


FIG. 10. PHYSICAL ARRANGEMENT IN THREE-POINT DVMC EXPERIMENT.

Consider the two VMC's associated with two given values $\gamma_o < \gamma_1$, of the bipolar parallax. Let $Q_o = (\gamma_o, \phi_o)$ and $Q_1 = (\gamma_o, \phi_1)$ be two points movable on the outer circle $\gamma = \gamma_o$ and let $Q_2 = (\gamma_1, \phi_2)$ be a freely adjustable point on the circle $\gamma = \gamma_1$ (Fig. 10). The observer is asked to leave Q_o and Q_1 fixed and to adjust the point Q_2 so that for the corresponding perceived points P_o, P_1, P_2 the sensed distance from P_1 to P_o equals the sensed distance from P_o to P_2.

As convenient and appropriate iseikonic coordinates for this experiment we take

$$\Gamma = \gamma - \gamma_o \quad \text{and} \quad \Phi = \phi - \phi_o$$

The visual coordinates (r, φ) are related to these by $r = r(\Gamma)$, $\varphi = \Phi$. Thus, with the understanding that $\omega = r(0)$, the visual coordinates of the points are defined by

$$P_o = (\omega, 0), \quad P_1 = (\omega, \varphi_1), \quad P_2 = (r, \varphi_2)$$

where $\varphi_1 = \phi_1 - \phi_o$, $\varphi_2 = \phi_2 - \phi_o$ and $r = r(\gamma_1 - \gamma_o)$. (See Fig. 11).

From the condition

$$D(P_o, P_1) = D(P_o, P_2) = d$$

and with the use of the cosine law, equation (9), we obtain

$$\cosh^2 d = \cosh^2\omega - \sinh^2\omega \cos \varphi_1 = \cosh r \cosh \omega - \sinh r \sinh \omega \cos \varphi_2$$

whence

$$\cos \varphi_1 = \frac{\sinh r \cos \varphi_2}{\sinh \omega} + \frac{\cosh^2\omega - \cosh \omega \cosh r}{\sinh^2\omega}$$

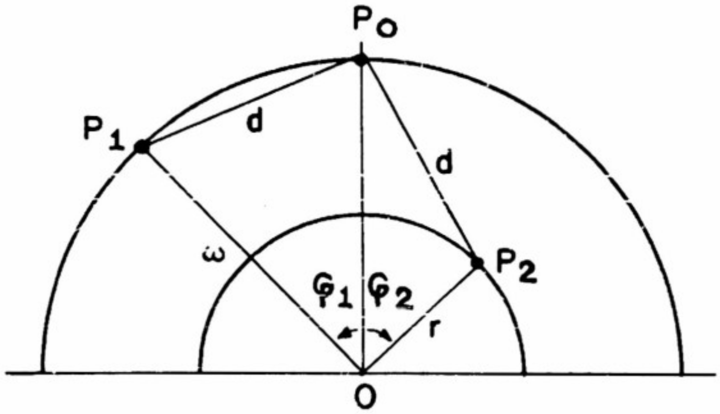


FIG. 11. REPRESENTATION OF THE SENSORY SITUATION IN THE THREE-POINT DVMC EXPERIMENT.

In the above equation put

(24)
$$m = \frac{\sinh r}{\sinh \omega}, \qquad b = \frac{\cosh^2\omega - \cosh\omega\cosh r}{\sinh^2\omega},$$
$$Y = \cos\varphi_1, \qquad X = \cos\varphi_2.$$

The quantities m and b are clearly constants depending only on the value $\Gamma = \gamma_1 - \gamma_0$ and not the particular values φ_1 and φ_2. Thus, if we repeat the experiment for different values of $\varphi_1 = \phi_1 - \phi_0$ and determine the corresponding values of $\varphi_2 = \phi_2 - \phi_0$, the plot of $\cos\varphi_1$ as ordinate against $\cos\varphi_2$ as abscissa will in theory be a straight line,

(25)
$$Y = mX + b.$$

It is an experimental fact that this graph is very nearly linear. Luneburg[14] has shown that if this result holds for each pair of Vieth-Müller Circles, then the space has constant curvature.

The values of m and b are easily determined from the plotted graph. The value of ω may then be found by eliminating r from the equations for m and b. Thus

$$\sinh^2 r = m^2\sinh^2\omega = \cosh^2 r - 1$$
$$\cosh^2 r = \cosh^2\omega - 2b\sinh^2\omega + \frac{b^2\sinh^4\omega}{\cosh^2\omega}.$$

Combining these equations and setting $\cosh^2\omega = \sinh^2\omega + 1$, we get

$$1 + m^2\sinh^2\omega = 1 + (1 - 2b)\sinh^2\omega + b^2\frac{\sinh^4\omega}{\cosh^2\omega}$$

whence,

$$m^2 = 1 - 2b + b^2\frac{\sinh^2\omega}{\cosh^2\omega} = 1 - 2b + b^2\left(1 - \frac{1}{\cosh^2\omega}\right)$$

and

(26)
$$\cosh^2\omega = \frac{b^2}{(1 - b)^2 - m^2}$$

Having determined ω in this fashion, the value of r is easily found from the equation (24) for m.

It is clear that the quantity on the left in (25) must be greater than 1 if ω is to be a real quantity. The fact that this is experimentally true is further evidence that the geometry is hyperbolic. It can be seen that the geometry is hyperbolic, euclidean, or elliptic, according to whether m^2 is greater than, equal to, or less than 1-2b.

(ii) *The Four-Point Experiment.* The three-point method is found to be somewhat insensitive since the values of X and Y in equation (25) are plotted upon points much nearer to each other than to the intercept of the line (see Part II, Fig. 25). It follows that the intercept b, depends rather critically on the determination of the slope m. In order to surmount this difficulty, Luneburg[14] suggested a method of determining m by the use of four points.

Let $Q_1 = (\gamma_o, \phi_1)$ and $Q_2 = (\gamma_o, \phi_2)$ be two points fixed on the circle $\gamma = \gamma_o$ and let $Q_3 = (\gamma_1, \phi_3)$ and $Q_4 = (\gamma_1, \phi_4)$ be two other points which slide on the circle $\gamma = \gamma_1$. Let P_1, P_2, P_3, P_4 be the corresponding sensed points. The observer is asked to equate the sensed distance $D(P_3, P_4)$ to $D(P_1, P_2)$.

Setting $r = r(\gamma_2 - \gamma_1)$, as before, and using

$$\Delta_1 = \phi_2 - \phi_1$$
$$\Delta_2 = \phi_4 - \phi_3$$

(Fig. 12) we obtain by the cosine law:

$$\cosh^2\omega - \sinh^2\omega\cos\Delta_1 = \cosh^2 r - \sinh^2 r \cos\Delta_2$$

whence,

$$\sinh^2\omega\,(1 - \cos\Delta_1) = \sinh^2 r\,(1 - \cos\Delta_2)$$

and

$$m^2 = \frac{\sinh^2 r}{\sinh^2\omega} = \frac{1 - \cos\Delta_1}{1 - \cos\Delta_2}.$$

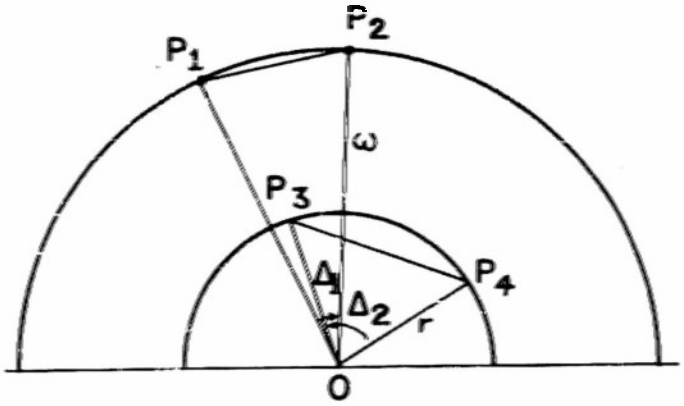


FIG. 12. REPRESENTATION OF THE SENSORY SITUATION IN THE FOUR POINT DVMC EXPERIMENT

From $1 - \cos\Delta = 2\sin^2 \frac{1}{2}\Delta$ we obtain, finally

(27) $$m = \frac{\sinh r}{\sinh \omega} = \frac{\sin \frac{1}{2}\Delta_1}{\sin \frac{1}{2}\Delta_2}$$

This value of m may then be used for a better determination of b in (25).

From the values of m and b we may then calculate the value of ω from (26). Once we obtain the value of ω experimentally we need no longer use the three-point experiment, but by repeated use of the four-point experiment with differing values of $\Gamma = \gamma_2 - \gamma_1$, we can calculate r (Γ) from (27).

5c. The Equipartitioned Parallel Alleys

This experiment is significant in that the calculation of perceived distance r is altogether independent of any of the hypotheses concerning the role of γ in the perception of distance. It has potential use, therefore, as a test of the degree of validity of these hypotheses.

The observer is asked to arrange six lights, three on each side of the median, so as to form a parallel alley as in Section 5a. Let the six lights be designated by the symbols $Q_i^{\pm} = (\gamma_i, \pm\varphi_i)$ (i = 1, 2, 3,). Let us suppose that $\gamma_1 > \gamma_2 > \gamma_3$. The two lights $Q_3^{\pm}$ are fixed in position. The lights $Q_1^{\pm}$, are restricted to motion on the VMC $\gamma = \gamma_1$. The

lights $Q_2^{\pm}$ are freely movable in the horizontal plane (Fig. 13).

The observer adjusts the lights $Q_1^{\pm}$ and $Q_2^{\pm}$ so that the corresponding sensed lights $P_1^{\pm}$ and $P_2^{\pm}$ appear to be lined up with $P_3^{\pm}$ in a parallel alley. The lights $Q_2^{\pm}$ are then further adjusted so that the observer perceives the points $P_2^{\pm}$ as being exactly midway in distance between $P_1^{\pm}$ and $P_3^{\pm}$. When this has been done we say the alley has been *equipartitioned*, or simply *partitioned*, and we refer to the points $Q_2^{\pm}$ as the partition points.

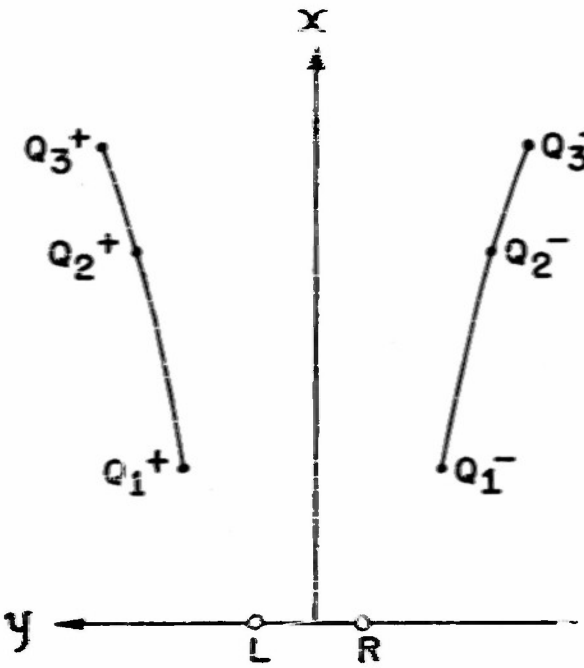


FIG. 13. PHYSICAL ARRANGEMENT OF AN EQUIPARTITIONED PARALLEL ALLEY

For this experiment the appropriate iseikonic coordinates are

$$\Gamma = \gamma - \gamma_2 \quad \text{and} \quad \Phi = \phi.$$

The equation of the parallel alleys, as we have already seen, is

(28) $\tanh r \quad \sin\varphi = \tanh\omega \quad \sin\varphi_3 = \tanh Y$

where Y is the radial distance from the origin of the point P_o on the alley at $\varphi = \pi/2$ (Fig. 14).

Set $X_i = D(P_o, P_i)$, $(i = 1, 2, 3)$. From the right-triangle law (13) we have

(29) $$\tan\varphi_i = \frac{\sinh Y}{\tanh X_i}, \quad (i = 1, 2, 3).$$

Since P_2 is perceived as being midway between P_1 and P_3 we have

(29a) $$X_2 = \tfrac{1}{2}(X_1 + X_3).$$

By employing (29) and (29a) together we find:

$$\tanh 2X_2 = \frac{2\tanh X_2}{1 + \tanh^2 X_2} = \frac{2\sinh Y \tan\varphi_2}{\tan^2\varphi_2 + \sinh^2 Y} = \tanh(X_1 + X_3) = \frac{\tanh X_1 + \tanh X_3}{1 + \tanh X_1 \tanh X_3} = \frac{\sinh Y(\tan\varphi_3 + \tan\varphi_1)}{\tan\varphi_3 \tan\varphi_1 + \sinh^2 Y}$$

Using this equation we determine Y by

(30) $$\sinh^2 Y = \tan^2\varphi_2 \left[\frac{2 - (S + T)}{(S + T) - 2ST}\right]$$

where

(30a) $$S = \frac{\tan\varphi_2}{\tan\varphi_1} \qquad T = \frac{\tan\varphi_2}{\tan\varphi_3}$$

The value of Y is determined, therefore, only from the measured values of ϕ. The values

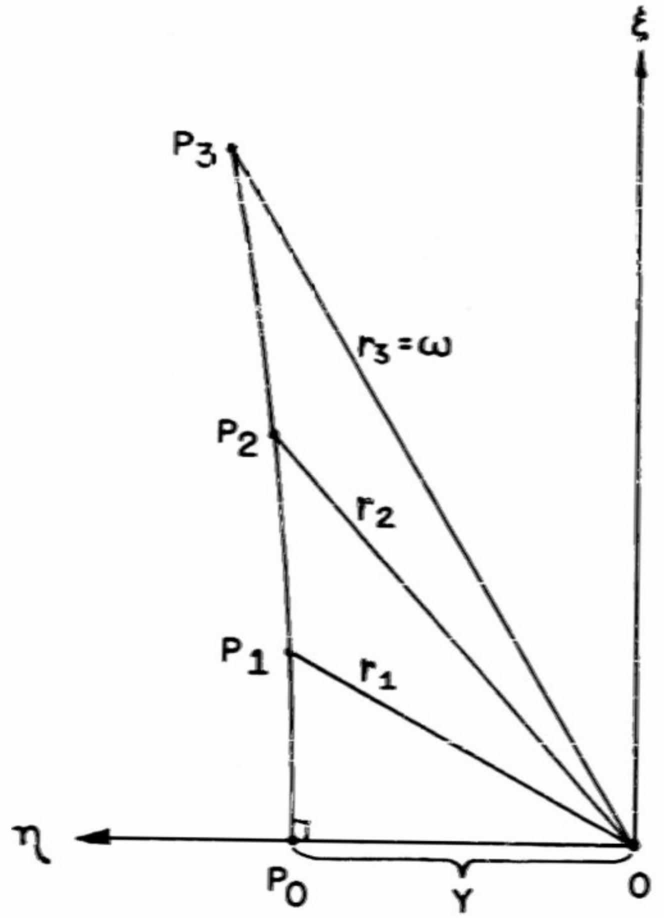


FIG. 14. REPRESENTATION OF THE SENSORY SITUATION CORRESPONDING TO FIG. 13.

of r are obtained from (28) and again only the values of ϕ are involved. In particular we have

(30b) $$\tanh \omega = \frac{\tanh Y}{\sin \varphi_3}$$

Clearly, the assumption $r = r(\Gamma)$ does not enter in the design of the experiment in any way. Since perceived distance as measured by this experiment is independent of any hypotheses concerning the nature of sensed equidistance, particularly H_1, H_2 and H_3, it may be used to test the validity of these assumptions. To do this in adequate detail would require in excess of 100 experiments per observer.

The equipartitioned alleys also give evidence that the space is hyperbolic. This is the consequence of the fact that the quantity on the right in (30) is found experimentally to be positive. If it were zero or negative we would take the result to mean that the geometry is euclidean or elliptic in the respective cases. It is easy to see that this condition amounts to saying

$$K \begin{smallmatrix}<\\=\\>\end{smallmatrix} 0 \quad \text{according to whether } \tfrac{1}{2}(\cot \varphi_1 + \cot \varphi_3) \begin{smallmatrix}<\\=\\>\end{smallmatrix} \cot \varphi_2 .$$

5d. Size Constancy – Relation of Perceived to Physical Size

The phenomenon of size constancy has received a great deal of attention in the literature (see C. H. Graham[15] for bibliography). The title alludes to the fact that the sizes of physical objects are not judged in proportion to the sizes of the retinal images but are generally seen more nearly in their correct physical relationships. In the performance of most tests of size constancy, clues such as perspective, the presence of familiar objects of known size, and other extraneous means of forming size judgments have generally been present. The study of the purely binocular basis for these judgments is another matter. In the experiment described here, size comparisons are made in the darkroom and care is taken to prevent extraneous information from reaching the observer.

The observer is asked to make the same sort of judgment as in the four-point experiment Section 5b (ii). The points $Q_1^{\pm}$ are fixed symmetrically to the median at $(\gamma_1, \pm\phi_1)$, and the observer sets the points $Q_2^{\pm}$ at some closer distance to give the impression of being symmetric to the median with $D(P_1^+, P_1^-) = D(P_2^+, P_2^-)$. Let us suppose that the points $Q_2^{\pm}$ are located at $(\gamma_2, \pm\phi_2)$. For iseikonic coordinates we use

$$\Gamma = \gamma - \gamma_1 ; \quad \Phi = \phi$$

The right-triangle law (13) gives the relation

$$\sinh r_1 \tan \varphi_1 = \sinh r_2 \tan \varphi_2 \tag{31}$$

where r_1, r_2 are the perceived radial distances of midpoints of the respective segments $\overline{P_1^+ P_1^-}$ and $\overline{P_2^+ P_2^-}$ (Fig. 15).

Using the approximation $\tan\phi = \frac{y\gamma}{2}$ we obtain the ratio of the two physical sizes from (31) as

$$\frac{y_2}{y_1} = \frac{\gamma_1 \sinh r_1}{\gamma_2 \sinh r_2} \tag{32}$$

If the size of the retinal image were the effective criterion, the ratio of the sizes would be

$$\frac{y_2}{y_1} = \frac{\gamma_1}{\gamma_2}$$

The departure from this ratio may be considered an indication of the effectiveness of our depth perception in judging the relative sizes of objects.

If the ϕ angles are sufficiently small we may use the approximations $r_1 \sim \omega$ and $r_2 \sim -(\gamma_2 - \gamma_1)$ to obtain

$$\frac{y_2}{y_1} \sim \frac{\gamma_1 \sinh \omega}{\gamma_2 \sinh r_2} \tag{32a}$$

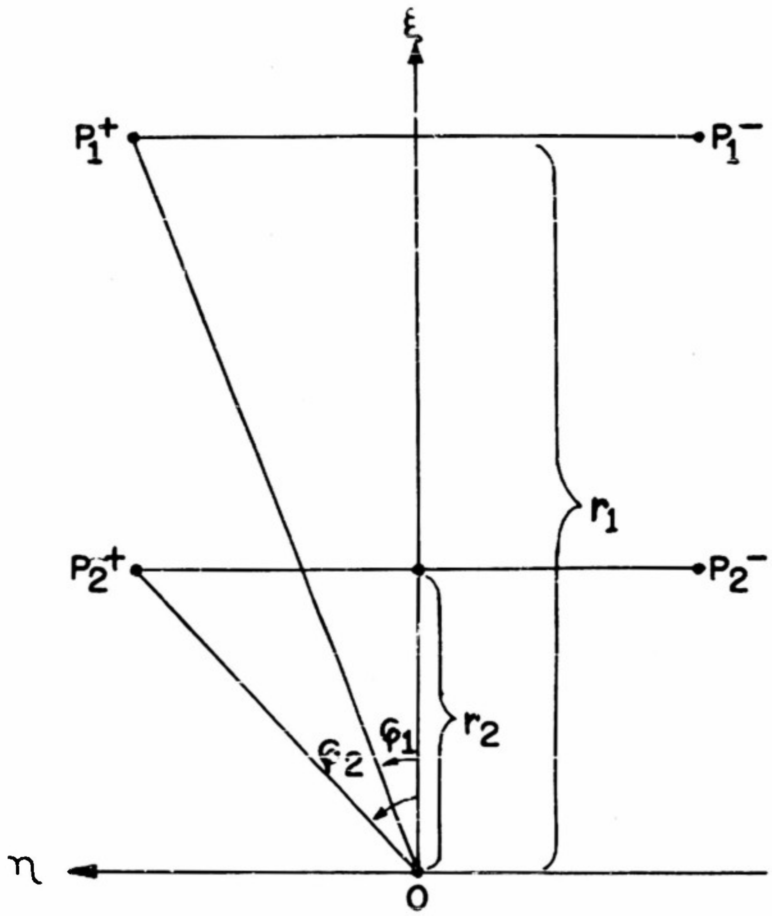


FIG. 15. REPRESENTATION OF THE SENSORY SITUATION IN MAKING A SIZE MATCH.

It should be stressed that the size-constancy relationship will depend upon the position of the distant reference object.

If we employ small values of ϕ we may use (32a) to determine r (Γ) once we know the value of ω. If we do not restrict ϕ in this way we should use equation (21) for the distance alleys instead.

Some results obtained in our laboratory do give evidence of size constancy, even for darkroom observation (C. J. Campbell[16]). The size constancy data alone cannot be utilized to demonstrate the curvature of visual space. However if the size constancy experiment were considered an equidistant alley and compared with a corresponding parallel alley, then the results could be used to determine the nature of the space in the manner of Section 5a.

5e. The Phenomenon of the Frontal Geodesics

To Helmholtz[17] we attribute the observation that the physically straights lines do not appear straight at all distances. Curves which do give the impression of straightness are not physically straight but are concave toward the observer at near distances and convex at

far (Fig. 16). For some intermediate distance the frontal geodesic will be straight in the vicinity of the median. Although this phenomenon is not very useful in computing r (Γ) it is an example of the kind of observation which may be given a quantitative description by means of the theory.

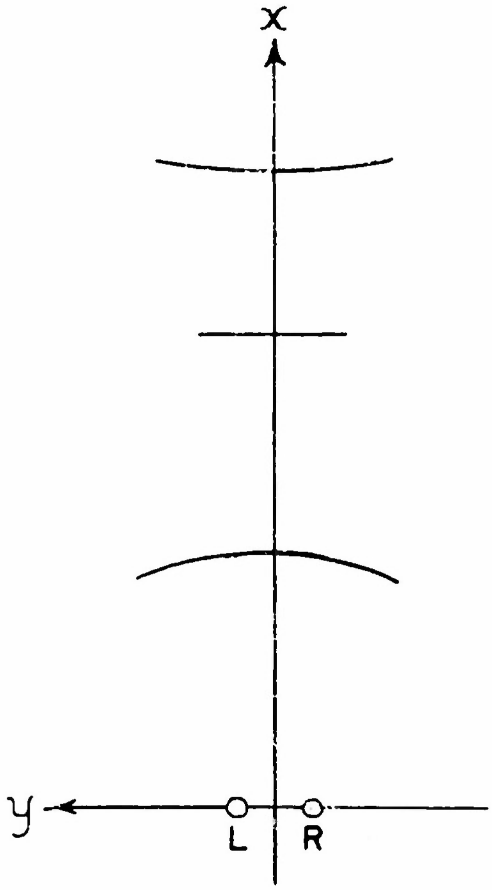


FIG. 16. COMPARISON OF FRONTAL GEODESICS SET AT DIFFERENT DISTANCES.

The equation of the frontal geodesics is easily written. Let us suppose we are dealing with the geodesic segment between the points $Q_o^{\pm} = (\gamma_o, \pm\phi_o)$. As iseikonic coordinates we take

$$\Gamma = \gamma - \gamma_o , \quad \Phi = \phi$$

The equation is then obtained from (11)

$$\tanh r \; (\Gamma) \; \cos\phi = \tanh\omega \; \cos\phi_o . \tag{33}$$

It may be of some interest to determine the distance of the straight frontal geodesic; i.e., the physical abscissa for which the frontal geodesic is physically straight. From the approximation (3a) we have for sufficiently small values of γ

$$x = \frac{2\cos^2\phi}{\gamma}$$

From (33), we have for the frontal geodesics

$$\cos^2\phi = \frac{\tanh^2\omega}{\tanh^2 r} \cos\phi_o .$$

Eliminating $\cos^2\phi$ we find

$$\gamma \tanh^2 r = \frac{2\tanh^2\omega\cos\phi_o}{x} .$$

If x is the distance of the straight geodesic, the term on the right is a constant in the neighborhood of the median. Hence, differentiating with respect to Γ, we obtain

$$\tanh^2 r + 2\gamma\frac{\tanh r}{\cosh^2 r}\frac{dr}{d\Gamma} = 0 .$$

Setting $\gamma = \frac{2}{x}$ for the point on the median we obtain

$$x = -\frac{8}{\sinh 2r_1} \cdot \frac{dr}{d\Gamma} \tag{34}$$

where r_1 is the perceived distance of the point on the median and the derivative is taken at $r = r_1$. If we approximate r_1 by ω we obtain

$$x \sim -\frac{8}{\sinh 2\omega}\left(\frac{dr}{d\Gamma}\right)_{\Gamma=0} \tag{34a}$$

Formula (34a) will be valid if the angle ϕ_o is not too large.

Having determined the function r (Γ) within experimental error, we shall be able to predict roughly the distance of the straight geodesic by (34a). Conversely, if we determine the position of the straight geodesic, we shall be able to reinforce our statistical knowledge of the function r (Γ) at the value $\Gamma = 0$ by determining the derivative in (34a).

6. SUMMARY

Luneburg's theory of binocular visual space rests upon the mathematical assumption that the visual space is a finitely compact and convex metric space [Section 3d (a) to (g)]. This statement means hardly anything more than the fact that observers are capable of making visual comparisons of length. Other assumptions (e.g. that the space is desarguesian and riemannian) lead to the conclusion that the geometry of visual space is one of the three simple geometries of constant gaussian curvature, either hyperbolic, euclidean or elliptic. Of the three alternatives, our experiments consistently support the first. The laws of the hyperbolic geometry of Bolyai and Lobachevski, therefore, most probably operate in the visual space.

If we designate coordinates r, φ, ϑ in the visual space corresponding respectively to radial distance, azimuth angle and angle of elevation we find that these quantities can be related to the physical coordinates of the stimulus configuration by means of the equations

$$r = r(\Gamma)$$
$$\varphi = \Phi$$
$$\vartheta = \Theta$$

where Γ, Φ, Θ are so-called iseikonic coordinates. To characterize an individual's response to geometrical spatial stimuli we have then only to determine the one function r (Γ). This is a feasible experimental project and several technics for effecting this determination are discussed. The description of three of these technics and the data derived from their use are given in Part II, Section 3.

7. CONCLUSION

Experimental evidence has given reason for a modification of some factors in the Luneburg theory by postulating a different mapping of physical into visual space. This modification, again, should not be considered as the final word in this matter, but only as an approximation

which is to be tested and improved by further experiment. Scattered throughout the discussion are suggestions as to possible fruitful courses of future experiment and more will occur to a reflective reader. Yet, even as it stands now, the theory is able to give a good qualitative (and to a considerable extent, quantitative) account of many of the geometrical phenomena of binocular space perception.

For its precise quantitative evaluation the theory must wait upon the detailed statistical evidence of a great many future experiments. Whatever the outcome of such an elaborate statistical study, it is felt that this kind of abstract geometrical approach will prove useful. The theory is held to be important as much (if not more) for its methods as for any specific results.

It should not be supposed that this modification in the analysis constitutes in any way a refutation of Luneburg's ideas. Luneburg always recognized that his suggested parameters were at best a working basis for experimental investigation. The same may be said for the modification, and we hope it will be possible for investigators to carry on with the extensive experimentation necessary to completely confirm the Luneburg theory.

PART II

THE EMPIRICAL SUPPORT OF THE LUNEBURG THEORY

In the first part of this report it was shown how it is possible to formulate a theory of binocular visual space in systematic fashion by logical deduction from the stated set of postulates. Some of these postulates are intuitively derived from our experience and can never be completely tested by experiment. Others which might feasibly be tested in the laboratory could not be explored in the time available for our program. Direct evidence was sought for certain hypothese such as the one concerning the perceptual role of the Vieth-Müller Circles (VMC), γ = constant, (Part I, Section 4, H_2). In the main, however, the tests of the theory have been based not so much upon the direct attack of its basic postulates as upon an investigation of their consequences. The test of the theory is whether it works.

The Luneburg theory *does* seem to work, - not perfectly with impeccable precision but well enough to be very significant. In one consistent account the theory succeeds in giving a description of a number of well-known binocular spatial phenomena which might, at first thought, appear to have no relation to each other. It also gives us a way of determining a quantitative relation between the visual and physical spaces. Although the quantitative aspects of the theory have not yet been placed upon a statistically firm footing, significant numerical results have been obtained. Further, the development of the theory is such that any consistent experimental deviation from an expected result can be utilized directly in definite ways to improve the theory.

In the following we shall see what light the experiments shed on the theory.

I. SENSED RADIAL DISTANCE

In Part I, Section 4b, the Vieth-Müller Circles (VMC), γ = constant, are ascribed the property of being perceived as loci of equal radial distance from the observer. To test this hypothesis, fifteen lights were set up in the horizontal plane adjustable along the ϕ - Lines, $\phi = 0°$, $\pm 5°$, $\pm 10°$, $\pm 15°$, ... $\pm 35°$ (Fig. 17). The light on the median, $\phi = 0°$, was fixed and the observer was asked to adjust the remaining lights according to the instruction:

> "The median light is fixed. Adjust the position of the other lights by having them moved toward you or away from you until you have the impression that, together with the median light, the lights form a circle about you with yourself at the center."

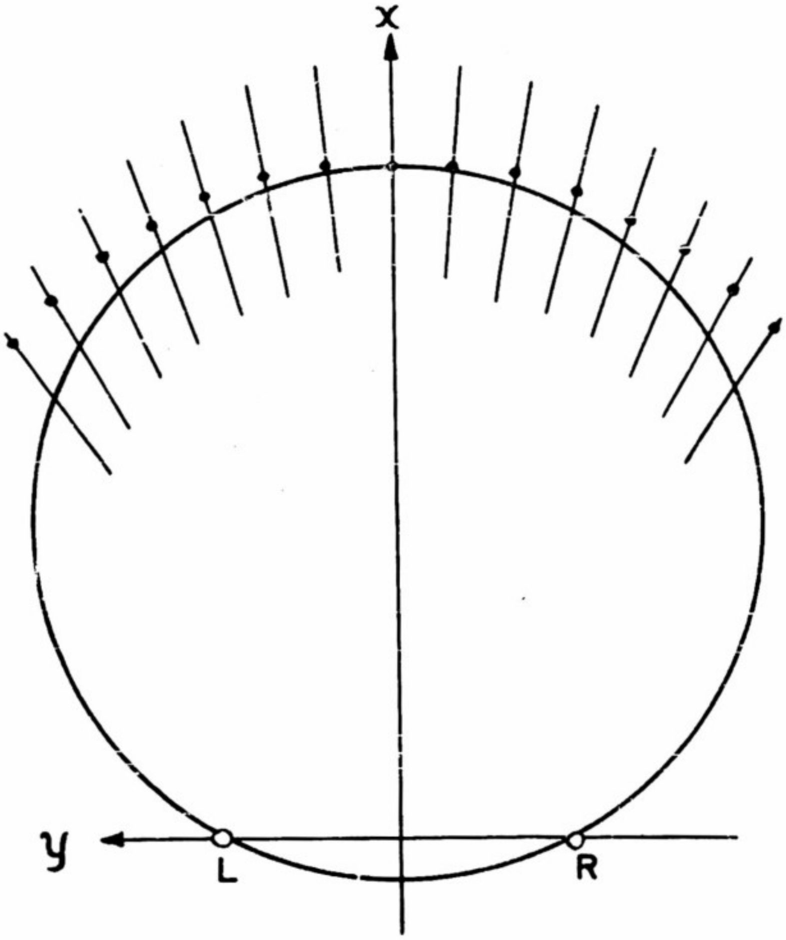


FIG. 17. SENSED RADIAL EQUIDISTANCE EXPERIMENT. PHYSICAL ARRANGEMENT OF THE LIGHTS.

In the limited number of experiments performed on several observers the lights do not always seem to fall on a VMC but, more generally, on a slightly flatter curve. The result of a sample experiment showing this type of deviation is given in Fig. 18. The effect decreases slightly with increasing distance. Occasionally an experimental setting actually fell inside the VMC. Since the experimental curve was close enough to the VMC in general to satisfy us with regard to use of the circle γ = constant as a first approximation, we did not pursue an extended course of experiments on this question. Furthermore there is a possibility that the flattening of the VMC may be attributable to experimental and theoretical factors such as the following:

(a) In the experimental situation, the lights were placed on a horizontal table covered with a sheet of coordinate paper so that their positions could be marked. Ordinarily, a great deal of attention was paid to keeping the illumination of the surroundings sufficiently low that the observer had no idea of the position of the lights with respect to the room. However, in this case with fifteen lights, although very dim, placed above a light reflecting surface, it is conceivable that the observer was able to obtain some shadowy impression of his surroundings and so would modify his setting of the lights to tend slightly toward the circle of equal physical distance.

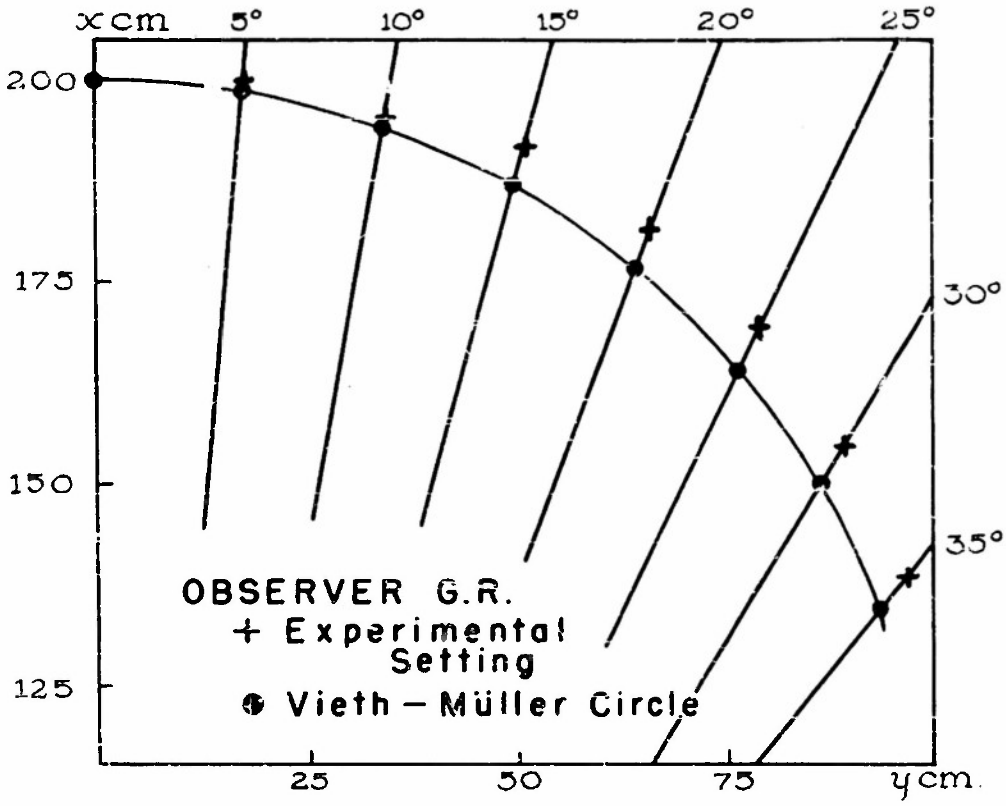


FIG. 18. SENSED RADIAL EQUIDISTANCE EXPERIMENT. TYPICAL SAMPLE SETTING. PLOTTED POINTS REPRESENT BILATERAL AVERAGE.

(b) The visual axis of the eye actually makes an angle with the optic axis at the anterior nodal point of approximately 5° temporally. If we define γ as the angle of convergence of the visual axes, the angle should presumably be measured with respect to the anterior nodal points. Since the position of the nodal points with respect to the head changes as the eyes shift fixation, this choice of coordinate is not as convenient as that based on rotation centers of the eyes. The use of the nodal points instead of the rotation centers does give a flatter curve than the VMC, but the effect predicted on this basis does not seem to be as great as the empirically determined flattening. The nodal points shift in the eye with accommodation also. Due to the drift of the nodal points in accommodation, the flattening should be most marked for the nearer VMC. However, the contribution of accommodation to this effect is minute and, although such an effect is found, it is quite likely attributable to the factor mentioned in (a).

If we were to make consistent use of the nodal points in defining the bipolar coordinates in three dimensions, it would be necessary to use Listing's Law to give the bipolar coordinates in terms of fixed physical coordinates. Since our experiments were conducted in the horizontal plane only, we have not felt the use of the nodal points instead of the centers of rotation would give sufficient advantage to justify the inconvenience.

For the purposes of the theory it is irrelevant whether or not we take the ocular mechanism into account in characterizing the loci of apparent equidistance. It would be sufficient to determine these loci experimentally and then to devise mathematically a suitable parametric representation for the experimental curves.

2. TESTS OF THE ISEIKONIC TRANSFORMATIONS

In this section we shall exhibit a considerable amount of evidence to show that binocular spatial relations are invariant under the iseikonic transformations (Part I, Section 4a). In other words, the perceptions of straightness, relative distance, form, etc. among the points of a stimulus configuration are not altered by changing the bipolar coordinates (γ, ϕ, θ) for each point of the configuration by fixed constant amounts. Since we have restricted ourselves to work in the horizontal plane, $\theta = 0$, we shall consider only special transformations of the form

$$\gamma' = \gamma + \lambda \qquad (35)$$
$$\phi' = \phi + \mu$$

A transformation of this kind may be subdivided into two separate transformations, one of the form

$$\gamma' = \gamma + \lambda \qquad (36)$$
$$\phi' = \phi$$

and the other of the form

$$\gamma' = \gamma \qquad (37)$$
$$\phi' = \phi + \mu$$

It will, therefore, be sufficient to treat each of the two special transformations separately rather than to work with the more general kind in which neither λ nor μ vanish.

2a. The Transformation $\phi' = \phi + \mu$, $\gamma' = \gamma$.

In this transformation the ϕ coordinates of the points of the stimulus configuration are are all changed by the same constant amount. The value of γ for each point is left fixed. Now, if binocular metric relationships are not changed by altering the stimulus in this manner, then the perception of straightness of line should not be altered. To test the special iseikonic transformation (37) the observer was first asked to arrange a set of lights so that they appear to lie on a straight line between two pre-set fixed lights symmetrically disposed about the median; i.e., to form a frontal geodesic. Then the fixed lights were re-set by changing the ϕ angles equally while not altering the values of γ, and the observer was asked to repeat the experiment for the new setting of the fixed lights; i.e., to form an oblique geodesic. If the observer placed the lights for the new setting to correspond to the old one through equation (37), the hypothesis of the iseikonic transformation would be verified for this special case. This procedure was called the *Predicted Oblique Geodesics Experiment.*

In the laboratory, nine lights Q_n were placed so as to be adjustable along the ϕ-lines,

$\phi_n = 5n°$, ($n = 0, \pm 1, \pm 2, \pm 3, \pm 4$). The fixed lights at $\phi = \pm 20°$ were pre-set symmetrically to the median at $x = 330$ cm. The observer adjusted the remaining lights according to the specific instruction:

> ''The two end lights are fixed. Adjust the remaining lights by having them moved toward you or away from you until they appear to lie on a straight line between the end points.''

The experiment was repeated several times under these conditions. With the mean of the repeated settings taken as the basis for computation, the total stimulus configuration was subjected to the transformation (37) with $\mu = +10°$. The point $Q_n = (\gamma_n, \phi_n)$ of the original configuration was then transformed into the point $Q'_n = (\gamma_n, \phi_n + 10°)$. This transformation replaced the original configuration with another stretching from Q'_{+4} at $\phi = +30°$ to Q'_{-4} at $\phi = -10°$. For lack of space we could utilize for experiment only the part of the configuration stretching from Q'_{+2} at $+20°$ to Q'_{-4} at $-10°$. Fixing two lights at Q'_{-4} and Q'_{+2} the experiment was repeated using the same instructions and with the lights placed at 5° intervals between $-10°$ and $+20°$. A similar series of observations was obtained also for $\mu = -10°$.

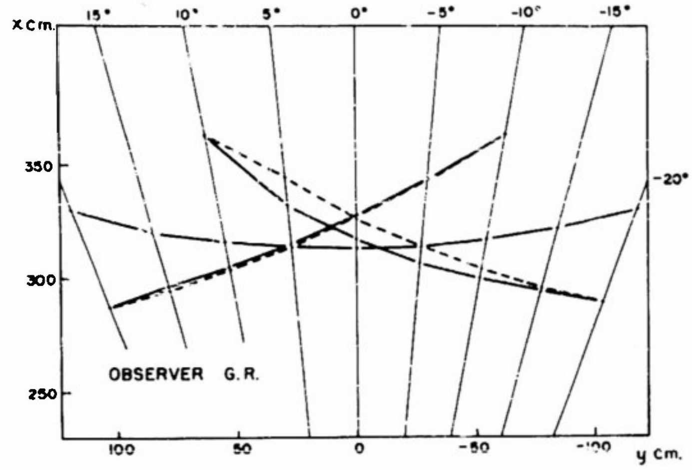
In Fig. 19 we compare the results of these settings for five observers with the predictions on the basis of the iseikonic transformation (37). The data are given in tabular form in Table I.

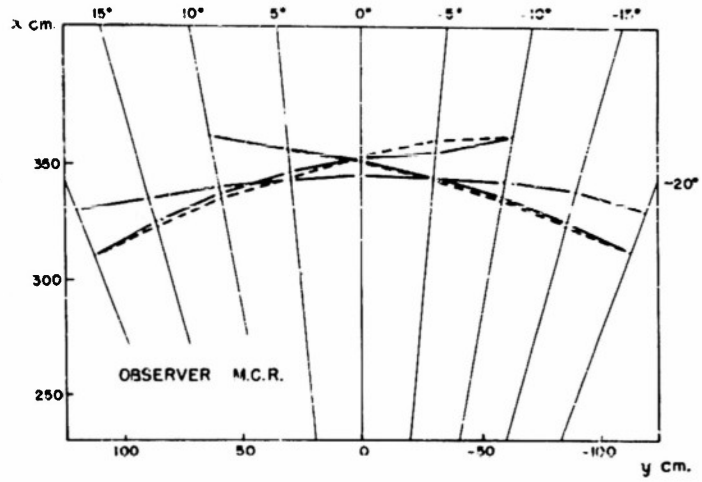
In general, the agreement between prediction and experiment is good. Wherever there is a marked deviation of the setting from the prediction there is also a marked asymmetry. For observers who exhibit this asymmetry we might reasonably assume that the two eyes do not play equal roles in binocular vision. The interesting problem of generalizing the theory for such observers is left open.

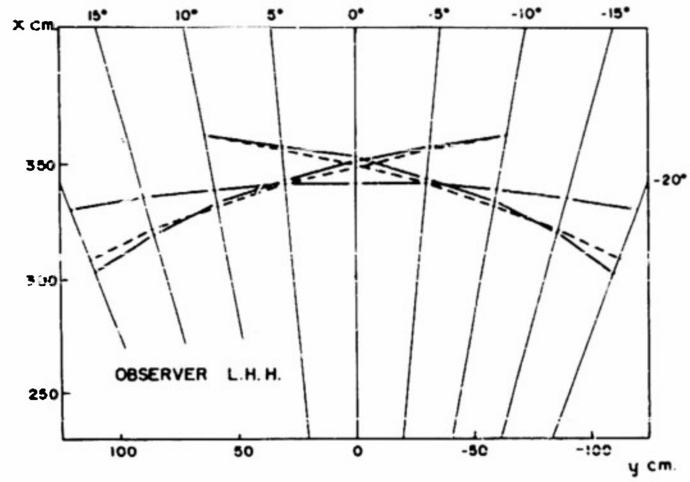
This experiment was actually designed to test a somewhat different hypothesis. For the present purpose it would have been desirable not to alter the number of points in the stimulus configuration so that the original and transformed configurations might be complete images of each other under iseikonic transformation. However, it is felt that the conditions were adequate to bring out the point in question.

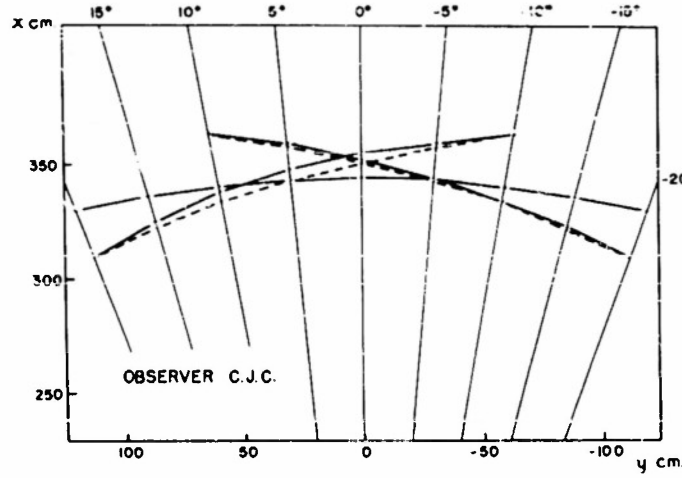
2b. The Transformation $\gamma' = \gamma + \lambda$, $\phi' = \phi$

The two experiments described in this section were not designed originally to test the invariance of binocular metric relationships under the transformation (36). They were to be used for the determination of the functional connection between the visual radial coordinate r and the convergence angle γ. However, because of the manner in which the experiments were









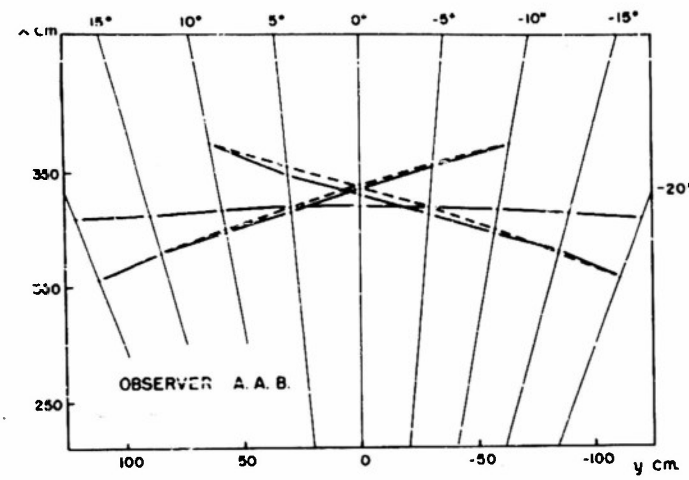


Fig. 19. Frontal and Oblique Geodesics. Solid lines show experimental settings. Dotted lines show prediction of oblique settings computed from frontal settings.

TABLE I

Test of the iseikonic transformation $\emptyset' = \emptyset + \mu$, $\tau' = \tau$ (Predicted Oblique Geodesics Experiment). Average settings of x in centimeters for each value of $\emptyset$ for a frontal geodesic and two oblique geodesics when the total stimulus configuration was subjected to the transformation $\emptyset' = \emptyset \pm 10°$. Values in italics are those predicted on the basis of the iseikonic transformation, computed from the settings of the frontal geodesic.

Geodesic	No. of experiments	Setting of x for $\emptyset$ values of 20°	15°	10°	5°	0°	-5°	-10°	-15°	-20°
		Observer G.R.								
Frontal	10	330.0	320.0	315.9	313.8	313.2	313.8	317.2	322.1	330.0
Oblique										
Predicted				362.5	340.4	325.7	313.8	303.8	295.0	288.8
Experimental	5			362.5	331.9	317.0	306.3	300.0	294.5	288.8
Oblique										
Predicted		287.6	295.0	303.8	313.8	327.1	342.7	362.5		
Experimental	5	287.6	296.6	304.0	314.5	327.4	343.0	362.5		
		Observer M.C.R.								
Frontal	10	330.0	335.1	340.5	343.6	345.3	344.3	343.0	339.2	330.0
Oblique										
Predicted				362.5	356.4	351.1	343.6	334.9	323.7	312.3
Experimental	4			362.5	356.4	352.0	344.4	336.4	325.2	312.3
Oblique										
Predicted		310.0	321.1	334.9	344.3	353.6	360.8	362.5		
Experimental	4	310.0	324.6	337.5	346.4	353.2	355.6	362.5		
		Observer L.H.H.								
Frontal	5	330.0	335.4	338.6	341.7	341.0	341.7	338.8	335.0	330.0
Oblique										
Predicted				362.5	356.8	349.1	341.7	331.3	321.2	308.4
Experimental	7			362.5	357.8	352.8	343.3	333.8	320.3	302.6
Oblique										
Predicted		308.3	321 2	331.3	341.7	349.2	356.8	362.5		
Experimental	7	* 302.8	319.0	332.7	342.4	351.1	357.6	362.5		
		Observer C.J.C.								
Frontal	10	330.0	335.7	340.0	342.8	343.8	342.5	339.7	335.5	330.0
Oblique										
Predicted				362.5	357.1	350.6	342.8	333.4	322.0	309.3
Experimental	5			362.5	358.7	351.7	343.5	333.4	322.9	309.3
Oblique										
Predicted		309.6	322.2	333.4	342.5	350.3	356.5	362.5		
Experimental	5	309.6	324.4	337.5	347.3	354.8	358.7	362.5		
		Observer A.A.P.								
Frontal	10	330.0	331.6	333.4	334.9	335.3	335.1	334.2	332.6	330.0
Oblique										
Predicted				362.5	352.7	343.8	334.9	325.2	315.0	304.3
Experimental	4			362.5	348.9	340.6	332.0	323.5	315.7	304.3
Oblique										
Predicted		303.6	314.9	325.2	335.1	344.6	353.8	362.5		
Experimental	4	303.6	314.7	324.2	333.1	343.5	353.0	362.5		

* In this case the end light was incorrectly set.

executed, they provide good tests of the stated invariance under the special transformation

$$\gamma' = \gamma + \lambda$$
$$\phi' = \phi$$

All the experimental results exhibited in this section were obtained by the use of special stereoscopic devices. The instrumentation will be discussed in Section 4.

For each type of experiment, however, some results were also obtained with lights viewed without this instrumentation, - that is, with lights viewed directly. The range of conditions that could be investigated in this way was limited by the available laboratory space, but the results were not notably different from those obtained with the stereoscopic devices.

2b (i) The Double Vieth-Müller Circles. Three-Point Experiment.

This experiment is the same as that described in Part I, Section 5b (i). The telestereoscopic device to be described in Section 4 was used. The observer is shown three lights Q_o , Q_1 , Q_2. The lights Q_o and Q_1 are placed on the VMC $\gamma = \gamma_1$, with Q_o on the median. The light Q_2 is restricted to move on the VMC $\gamma = \gamma_2$ with $\gamma_2 > \gamma_1$ (see Part I, Fig. 10). The observer is asked to make a setting according to the specific instructions:

> ''Three lights are presented to you. Two of them are fixed in position and the third can be moved.'' (This light and its range of motion are demonstrated.) ''Direct the experimenter to adjust this light so that the distance between it and the middle light appears to be the same as the distance between the pair of fixed lights. Allow the eyes to roam freely both ways over the spatial interval between each pair of lights. Be sure to fixate on each light in turn and to sweep the eyes across the interval between each pair of lights until you are satisfied that the two distances appear to you to be the same.''

A series of these experiments was undertaken with values of γ_1 ranging from zero to .07 and $\gamma_2 = \gamma_1 + .01$. In each case settings were taken for the same fixed sequence of values of ϕ_1, the azimuth angle associated with Q_1. If the hypothesis of the iseikonic transformations is correct, the values of the azimuth angle ϕ_2 of Q_2 associated with a given ϕ_1 should exhibit no marked trend as γ_1 increases but rather should fall randomly in the neighborhood of some central value. That this is actually the case may be seen from Table II and Fig. 20 which present the results of an extended series of observations for two observers.

The entire experimental series was performed twice and each entry in the table gives the mean value of ϕ_2 for the two series. The value of ϕ_2 in each series was computed as the average of three or four experimental settings. Each entry in the table thus represents at least six experimental observations.

TABLE II

Test of the iseikonic transformation $\gamma' = \gamma + \lambda$, $\phi' = \phi$ (Three-Point Double Veith-Müller Circle Experiment) Average settings of ϕ_2 for different values of γ_1 and ϕ_1, when $\Gamma = .01$. ϕ_1 and ϕ_2 are expressed in radians.

Value of ϕ_2 for different values of γ_1 and ϕ_1

ϕ_1 \ γ_1	.00	.01	.02	.03	.04	.05	.06	.07	Average value of ϕ_2
					Observer G. R.				
.1226	.0443	.0405	.0424	.0436	.0316	.0332	.0400	.0400	.0394
.1415	.0629	.0600	.0620	.0696	.0458	.0548	.0548	.0583	.0585
.1583	.0719	.0798	.0794	.0892	.0629	.0768	.0721	.0800	.0765
.1734	.0985	.0965	.1025	.1054	.0827	.0919	.0858	.1008	.0955
.1874	.1130	.1177	.1116	.1186	.0933	.1054	.1094	.1110	.1100
.2003	.1301	.1314	.1283	.1314	.1167	.1211	.1226	.1238	.1257
.2125	.1428	.1536	.1488	.1508	.1346	.1470	.1412	.1372	.1445
					Observer M.C.R.				
.1226	.0458	.0551	.0332	.0387	.0412	.0374	.0424	.0494	.0429
.1415	.0664	.0693	.0520	.0533	.0587	.0562	.0578	.0603	.0592
.1583	.0790	.0803	.0625	.0675	.0790	.0690	.0739	.0617	.0716
.1734	.1002	.0944	.0803	.0831	.0922	.0872	.0855	.0800	.0879
.1874	.1167	.1066	.1002	.0970	.1061	.1049	.1096	.0851	.1033
.2003	.1270	.1275	.1032	.1175	.1246	.1326	.1213	.1175	.1214
.2125	.1424	.1435	.1564	.1403	.1445	.1510	.1447	.1326	.1444

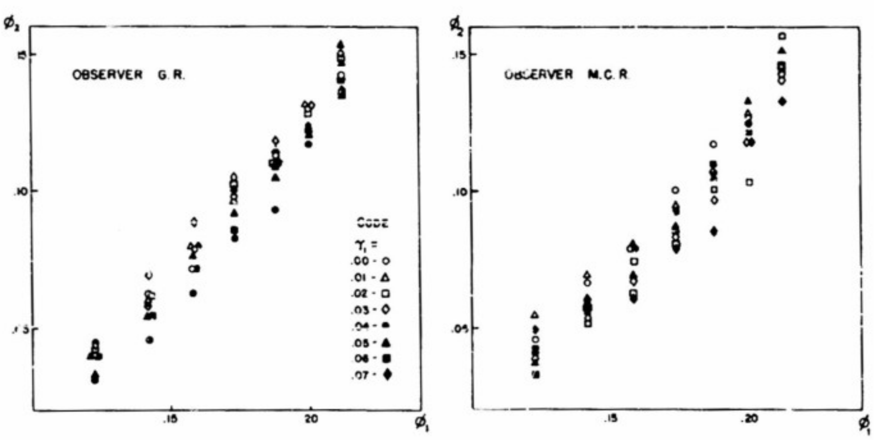


Fig. 20 Three-Point DVMC Experiment. Settings of ϕ_2 for each value of ϕ_1 and Γ when $\Gamma = .01$.

2b (ii) The Equipartitioned Parallel Alleys.

This experiment is the same as that discussed in Part I, Section 5c. The polaroid rack to be described in Section 4 was used. Six lights, $Q^{\pm}_n$, $(n = 1, 2, 3)$ are set out in two rows of three on either side of the median (see Part I, Fig. 13). The lights Q^{+}_3 and Q^{-}_3 are fixed symmetrically to the median at the respective points (γ_3, ϕ_3) and $(\gamma_3, -\phi_3)$. The lights Q^{+}_1 and Q^{-} are restricted to move on a line $x = x_1$. The remaining pair Q^{+}_2 and Q^{-}_2 may be moved freely in the two dimensions of the horizontal plane. A seventh light is placed on the median in line with Q^{+}_3 and Q^{-}_3 to aid the observer in establishing his orientation. The observer is asked to set the two rows of three lights in a parallel alley and then to set the middle light in each row exactly half way between the near and far lights. His specific instructions are these:

''The three distant lights are fixed. We shall call the central one the median light.

(1) Arrange the two rows of three lights on the right and left of the median so that they appear to you straight and parallel. Make sure that

(a) the two lines of lights have the same direction and that this direction appears parallel to that in which the median lights lies;

(b) the two lines appear to you perpendicular to the frontal plane, and

(c) the two lines appear to neither converge nor diverge in the distance. (Avoid the effect given by railroad tracks.)

(2) In each line of lights place the light intermediate between the near and far lights so that it appears exactly half way between the two.''

Some observers have difficulty in making a distinction between sensory parallelism and the impression given by physically parallel lines which most would agree is not one of sensory parallelism (e.g., the impression given by railroad tracks.)

In Table III and Fig. 21 we give the results of an experimental series for three observers. The value of ϕ_3 was fixed with $\tan \phi_3 = .1000$. Four values of γ_3 were used in equal steps ranging from about -.02 to +.02. The value of $\Gamma_1 = \gamma_1 - \gamma_3$ was approximately .039.

Four settings of the equipartitioned parallel alley were made by each observer for each choice of γ_3. For each γ_3, the mean of the four settings was taken and averaged again on the left and right. The values of $\tan \phi$ and Γ for the average settings are presented in Table III and Fig. 21. It is clear from the data that $\tan \phi$ does not show variation with γ_1 and depends

TABLE III

Test of the iseikonic transformation $\gamma' = \gamma + \lambda$, $\phi' = \phi$ (Equipartitioned Parallel Alley Experiment). Values of $\tan \phi_2$, $\tan \phi_1$, Γ_2 and Γ_1 for different positions of γ_3, when $\tan \phi_3$ = .1000.

Observer G.R.

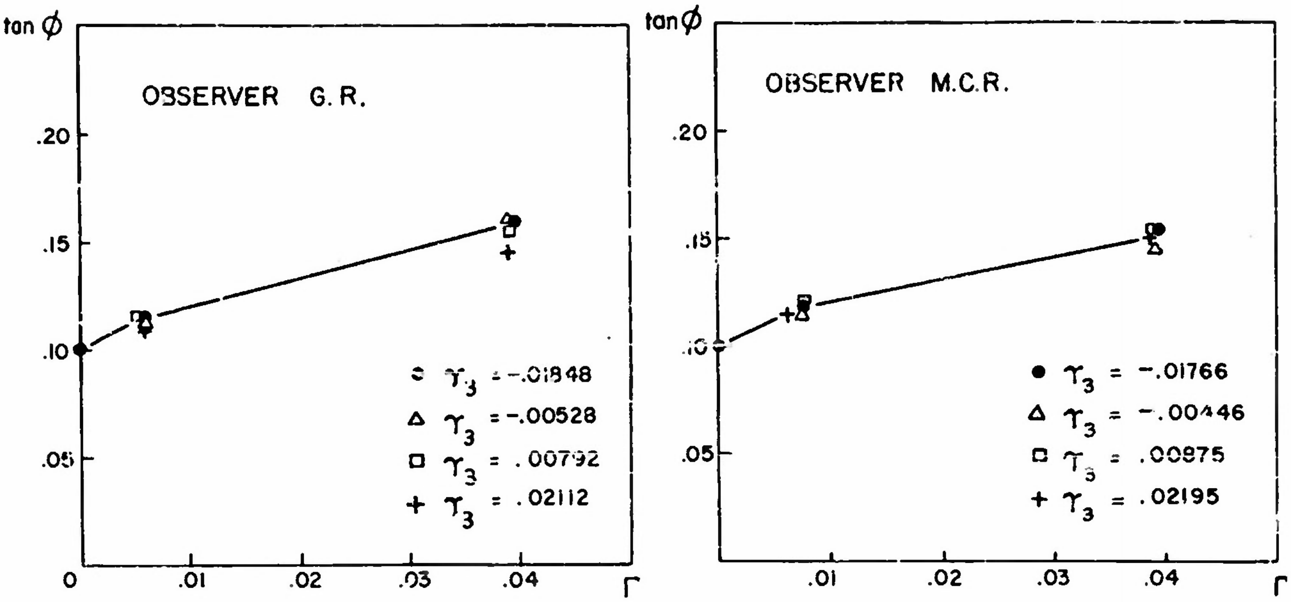
	$\gamma_3 = -.01848$	$\gamma_3 = -.00528$	$\gamma_3 = .00792$	$\gamma_3 = .02112$
$\tan \phi_2$	.1161	.1154	.1169	.1121
$\tan \phi_1$	.1604	.1617	.1583	.1451
Γ_2	.00593	.00572	.00497	.00585
Γ_1	.03928	.03906	.03891	.03892

Observer M.C.R.

	$\gamma_3 = -.01766$	$\gamma_3 = -.00446$	$\gamma_3 = .00875$	$\gamma_3 = .02195$
$\tan \phi_2$	.1188	.1155	.1192	.1142
$\tan \phi_1$	.1549	.1480	.1545	.1495
Γ_2	.00777	.00745	.00781	.00620
Γ_1	.03931	.03920	.03895	.03886

Observer C.J.C.

	$\gamma_3 = -.01848$	$\gamma_3 = -.00528$	$\gamma_3 = .00792$	$\gamma_3 = .02112$
$\tan \phi_2$	.1127	.1119	.1156	.1133
$\tan \phi_1$	.1475	.1415	.1531	.1497
Γ_2	.00684	.00678	.00685	.00802
Γ_1	.03906	.03926	.03898	.03887



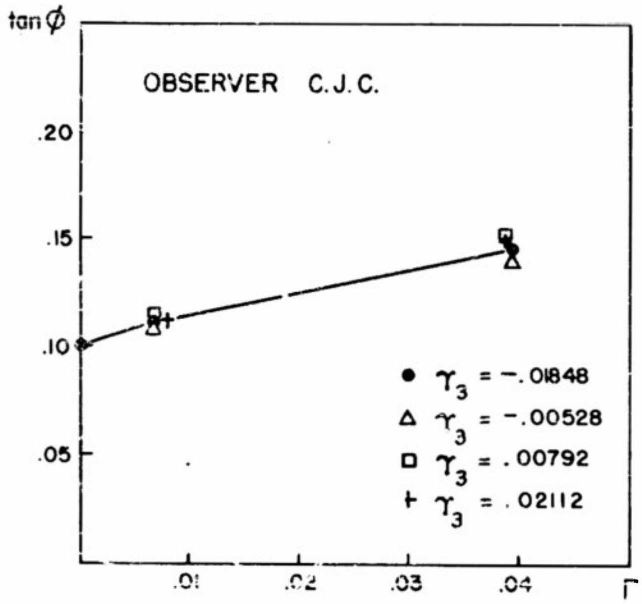


Fig. 21. Equipartitioned Parallel Alley Experiment. Plot of tan ϕ against Γ for four values of Υ_3, when tan ϕ_3 = .1000

only upon Γ. In this experiment also the assumption about the iseikonic transformations is confirmed for the special transformation (36) by the clear agreement among the data obtained under differing initial conditions.

3. DETERMINATION OF r (Γ)

In Part I, Section 4, we showed how it was possible to describe an individual's visual metric space in terms of a single function r (Γ). It then becomes a matter of considerable importance to obtain a good estimate of the values of this function. The first three experimental technics given in Part I, Section 5, were used for this determination. Since in each of these experiments small variations in an observer's settings may result in considerable differences in the values of the function, human variability becomes an important factor to consider.

The responses required of the observer were unusual and difficult. A tendency for the settings to drift in one direction was now and then noted, particularly when a new type of observation was initiated, but the settings soon stabilized. Random variation from day to day was also noted. Time did not permit us to study this factor in any detail. On the whole, considering what was required of the observer, we were surprised at the consistency of his settings. In a series of observations in which the effect of a progressive modification in the conditions was to be measured, we learned that a presentation of the individual experiments in random order resulted in a more stable picture of the effect to be measured and minimized any directional trend due to practice.

We realize that to obtain a meaningful estimate of a given individual's typical binocular spatial response it would probably be necessary to conduct a detailed statistical study with each experimental technic. Despite the fact that time did not permit us to make such an extended series of observations, the function r (Γ) emerges more clearly than might be expected. In general character, r is a monotonically decreasing function with a monotonically increasing slope. The values of the function for the two observers who have been able to complete the whole series of tests appear to be determined to within one part in five. Furthermore, individual differences are brought out. For one of these observers the values of r (Γ) are consistently somewhat higher than for those of the other. A greater r (Γ) may be interpreted as a greater absolute curvature of visual space. This greater curvature is not to be interpreted as a disadvantage. According to Luneburg[18], the greater the curvature of one's visual space, the more nearly will sensory matches of size approach a physical match.

3a. Parallel and Distance Alleys (The Blumenfeld Alleys)

In this experiment twenty lights were placed in two rows of ten on either side of the

median. The lights were arranged pairwise at each side of the median, both lights of a pair being freely and independently movable along a line x = constant. The ten chosen values of x are given in Table IV. The farthest distant lights were fixed at x = 500 cm., y = 35 cm. The observer was asked to set the lights in a parallel alley and then in a distance alley according to instructions which were in general similar to those used by Blumenfeld. They are as follows:

> ''This is an experiment dealing with space perception. We know one does not always perceive objects in space where they actually are in physical space, or to be of their actual physical size. We want to measure some of these differences.
>
> ''In the first experiment we shall show you some small lights which we shall arrange under your direction so that when you look down between them they appear to you to form straight, parallel lines of light. We wish you to think not of where the lights actually are, but merely of how you sense them. When they are all arranged, we want you to be able to say that these straight lines of lights as you see them could never, if extended, meet at any distance in front of you or at any distance behind you; that is, that they form walls that appear to you as parallel walls that appear neither to converge nor to diverge''

To familiarize the observer with the observation, a trial run utilizing only stations 1, 3, 5 and 8 was made, no measurements being taken of this trial. The instructions continued:

> ''In the second experiment we shall give you two pairs of lights at a time. The position of one pair will be fixed. We want you to direct us to move the lights of the other pair so that the lateral distance between them appears to you the same as that between the first pair. We want you to make an immediate, instantaneous judgment of whether the distance between the lights of the second pair is greater or smaller than or equal to that between the first pair. Do not think in terms of physical units of distance between the lights, for example, inches or centimeters. Just direct us in adjusting them until you immediately sense the two pairs of lights as being the same distance apart.''

Again a trial run utilizing only stations 1 and 3, 1 and 5, and 1 and 8 was made, again without measurements. After these preliminary observations, the experiment continued with the formation of the complete parallel and distance alleys in that sequence. The usual procedure was to give a second trial of each alley on the same day and to repeat the series on a second day.

The data given in Table IV and Fig. 22 for two observers represent the average of three such settings, averaged again on the left and right.

TABLE IV

Blumenfeld Parallel and Distance Alleys. Average setting of y for each value of x. Values of x and y are expressed in centimeters.

	Observer G.R.		Observer M.C.R.	
x	Parallel y_p	Distance y_d	Parallel y_p	Distance y_d
50	9.95	23.75	13.85	20.60
65	10.65	23.05	14.40	21.50
83	11.55	23.90	15.40	21.80
108	12.90	24.25	16.75	21.25
139	14.55	23.85	18.10	21.85
180	16.65	22.95	19.75	22.65
232	19.35	23.75	21.45	24.50
300	23.35	26.10	24.40	27.10
387	28.85	29.95	28.80	29.10
500	35.00	35.00	35.00	35.00

For each point of the setting the values of γ and ϕ were computed from the formulas

$$\tan\phi = y/x \qquad\qquad \gamma = p\,\frac{\cos^2\phi}{x}$$

(cf. equation [3a]) where p represents the distance between the rotation centers of the observer's eyes.* Letting (γ_1, ϕ_1) denote the bipolar coordinates of the most distant point, we then calculated $S = \sin^2\phi_d - \sin^2\phi_1$ for each point of the distance alley, $T = \sin^2\phi_p - \sin^2\phi_1$ for each point of the parallel alley and plotted S and T separately against $\Gamma = \gamma - \gamma_1$. An example of such a plot is given in Fig. 23. A curve was then drawn between the points of the plot, and values for S and T were taken from the curve at Γ = .02, .03, ... The ratio S/T was calculated for each value of Γ. The average value of the ratio weighted with respect

* THIS WAS ESTABLISHED APPROXIMATELY BY MEASURING THE INTERPUPILLARY DISTANCE.

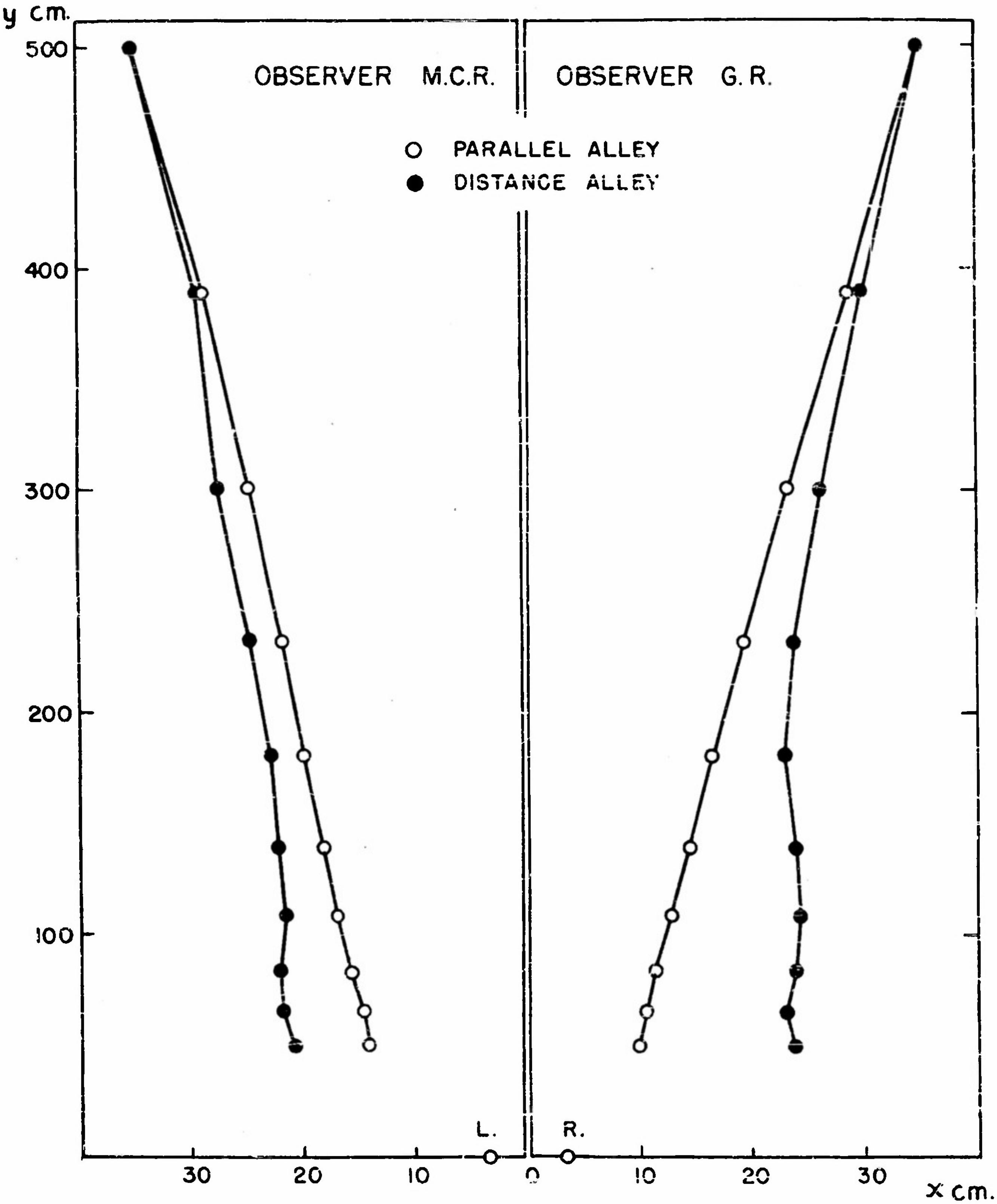


Fig. 22. Blumenfeld Parallel and Distance Alleys.

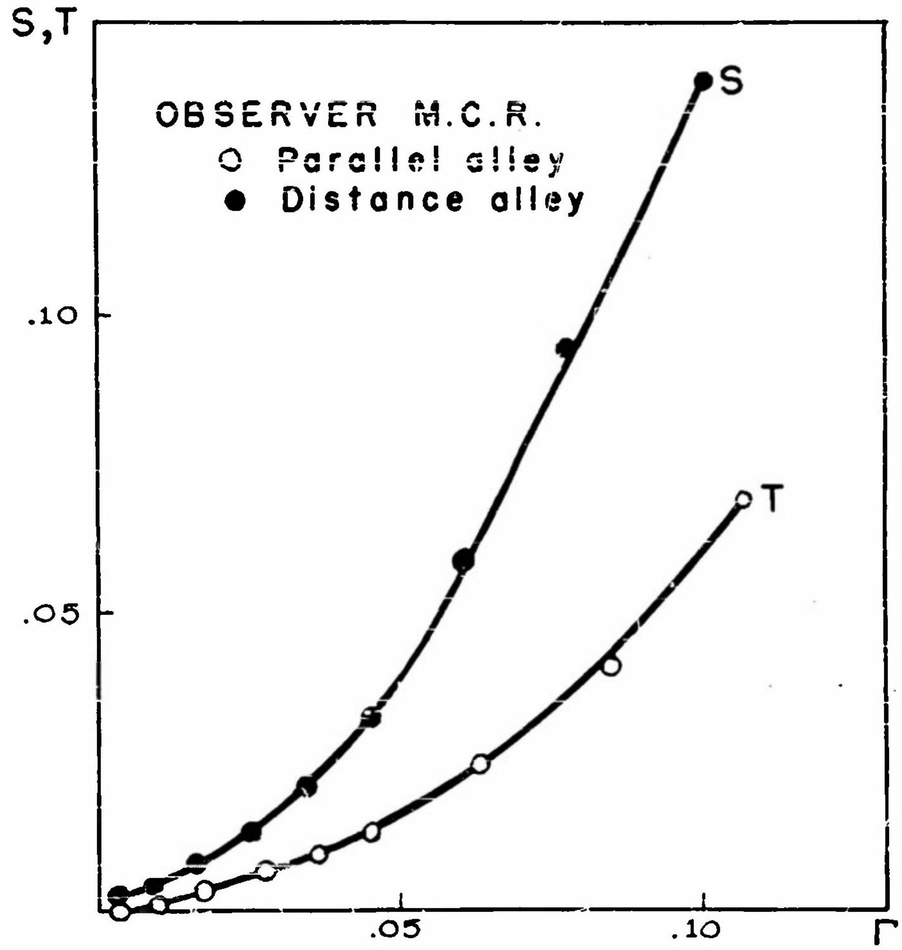


FIG. 23. BLUMENFELD PARALLEL AND DISTANCE ALLEYS. PLOTS OF S AND T AGAINST Γ.

to Γ was then determined.* The value of ω was found from this average by using the formula (23)

$$\cosh^2\omega = S/T$$

From the value of ω the function r (Γ) was then determined for each point of the setting by using the relation (20)

$$\tanh r = \tanh \omega \frac{\sin \phi_1}{\sin \phi_p},$$

for the points of the parallel alley; and the relation (21)

$$\sinh r = \sinh \omega \frac{\sin \phi_1}{\sin \phi_d},$$

for the points of the distance alley. The values of r (Γ) found by this method are presented

* WEIGHTED AVERAGE $= \dfrac{\sum \frac{S}{T} \Gamma}{\sum \Gamma}$

TABLE V

Twenty values of r (Γ) computed from the results of the Blumenfeld Parallel and Distance Alley Experiment.

Observer G. R. $\omega = 1.470$				Observer M. C. R. $\omega = .931$			
Parallel Alley		Distance Alley		Parallel Alley		Distance Alley	
Γ	r	Γ	r	Γ	r	Γ	r
.1105	.334	.0917	.329	.1103	.194	.1005	.195
.0832	.412	.0747	.418	.0843	.240	.0790	.236
.0629	.492	.0585	.499	.0642	.288	.0617	.290
.0457	.590	.0437	.617	.0469	.346	.0460	.378
.0328	.698	.0320	.772	.0338	.418	.0334	.465
.0225	.833	.0222	.975	.0233	.517	.0231	.568
.0147	.986	.0146	1.145	.0152	.625	.0151	.662
.0085	1.126	.0084	1.280	.0088	.741	.0088	.757
.0037	1.238	.0037	1.381	.0038	.844	.0038	.880
.0000	1.470	.0000	1.470	.0000	.931	.0000	.931

in Table V for two observers. The graphic representation of r plotted against Γ is displayed in Fig. 28 where these results are compared with those of other experiments.

3b. The Double Vieth-Müller Circle Experiments

The Double Vieth-Müller Circle experiments were performed with the lights viewed directly; i.e., without the teleostereoscopic device. The theoretical background for these experiments is given in Part I, Sections 5b (i) and 5b (ii).

Both the three-point and four-point experiments were performed for the four values of $\Gamma = \gamma_2 - \gamma_1 = .005, .01, .02, .04$. The convergence angle γ_1 was fixed throughout at .025 for observer G.R. and .026 for M.C.R. At any sitting, a mixed order of Γ and ϕ values was presented to the observer.

3b (i) The Three-Point Experiment. The observer was given the same instructions for the experiment as described in Section 2b (i). For ease in computation a slightly different technic was used. The lights $Q_o = (\gamma_1, 0)$ and $Q_2 = (\gamma_2, \phi_2)$ were left fixed and the light $Q_1 = (\gamma_1, \phi_1)$ was moved on the circle $\gamma = \gamma_1$, to satisfy the instruction of Section 2b (i). Settings of Q_1 were taken for five positions of Q_2 at $\phi_2 = 5°, 10°, 15°, 20°$ and $25°$. At least three settings were taken for each position of Q_2 at a given time. The entire series of experiments for the four values of Γ was performed twice.

TABLE VI

Three-Point Double Vieth-Müller Circle Experiment.
Average values of $Y = \cos \emptyset_1$, for given values of $X = \cos \emptyset_2$ and Γ.

Observer G.R. γ_1 = .025

Average values of Y for given values of X and Γ

X \ Γ	.005	.01	.02	.04
.9962	.9892	.9799	.9534	.8842
.9848	.9778	.9702	.9514	.8836
.9659	.9604	.9546	.9389	.8811
.9397	.9364	.9305	.9032	.8680
.9063	.9012	.8992	.8859	.8566
m =	.9613	.9002	.8995	.3486
b =	.0315	.0840	.0643	.5105
γ_0 =	.9928	.9842	.9638	.8891

Observer M.C.R. γ_1 = .026

Average values of Y for given values of X and Γ

X \ Γ	.0052	.0104	.0208	.0416
.9962	.9899	.9839	.9485	.8908
.9848	.9799	.9706	.9422	.8840
.9659	.9653	.9562	.9306	.8782
.9397	.9412	.9367	.9133	.8688
.9063	.9154	.9120	.8948	.8564
m =	.8447	.7737	.6135	.3668
b =	.1486	.2102	.3378	.5239
γ_0 =	.9933	.9839	.9513	.8907

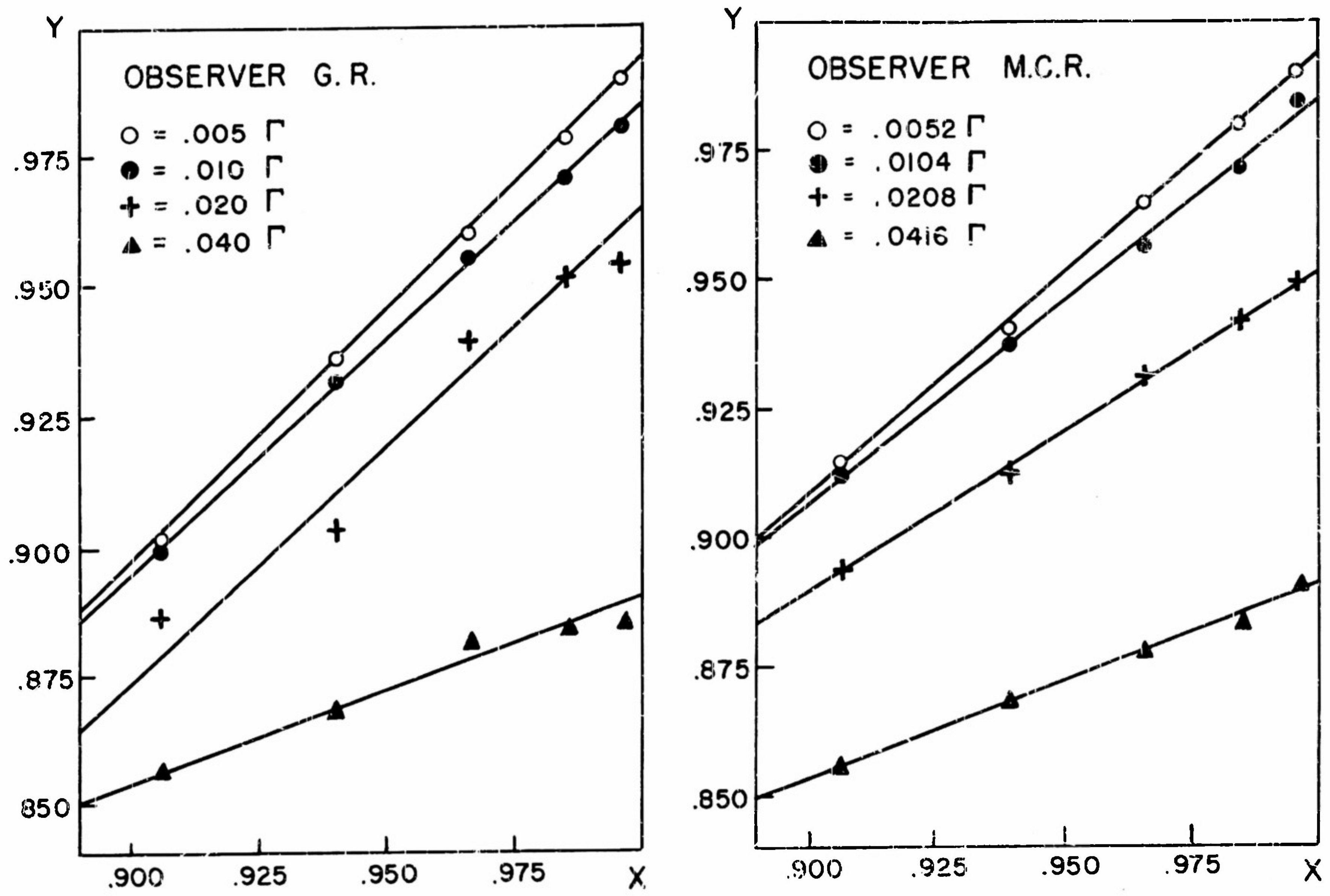


Fig. 24. Three-Point DVMC Experiment. Plot exhibiting linear relationship between X (cos ϕ_1) and Y (cos ϕ_2).

The values of $Y = \cos\phi_1$ were determined for each of the values of $X = \cos\phi_2$ (see equation [24]) at a given value of Γ, and then Y was plotted against X. In Table VI and Fig. 24 we show the average result of the entire experimental series for the four values of Γ. These results clearly exhibit the linear dependence of Y upon X for a given value of Γ. As Luneburg demonstrated, this linearity is evidence for the homogeneity of the geometry. In other words, the geometry must be one of the three types of constant curvature.

Using the representation of the line in the slope intercept form

$$Y = mX + b$$

we find that the three-point experiment is not sensitive with respect to the value of m and hence large variations in the calculated values of b will occur from time to time (Fig. 25).

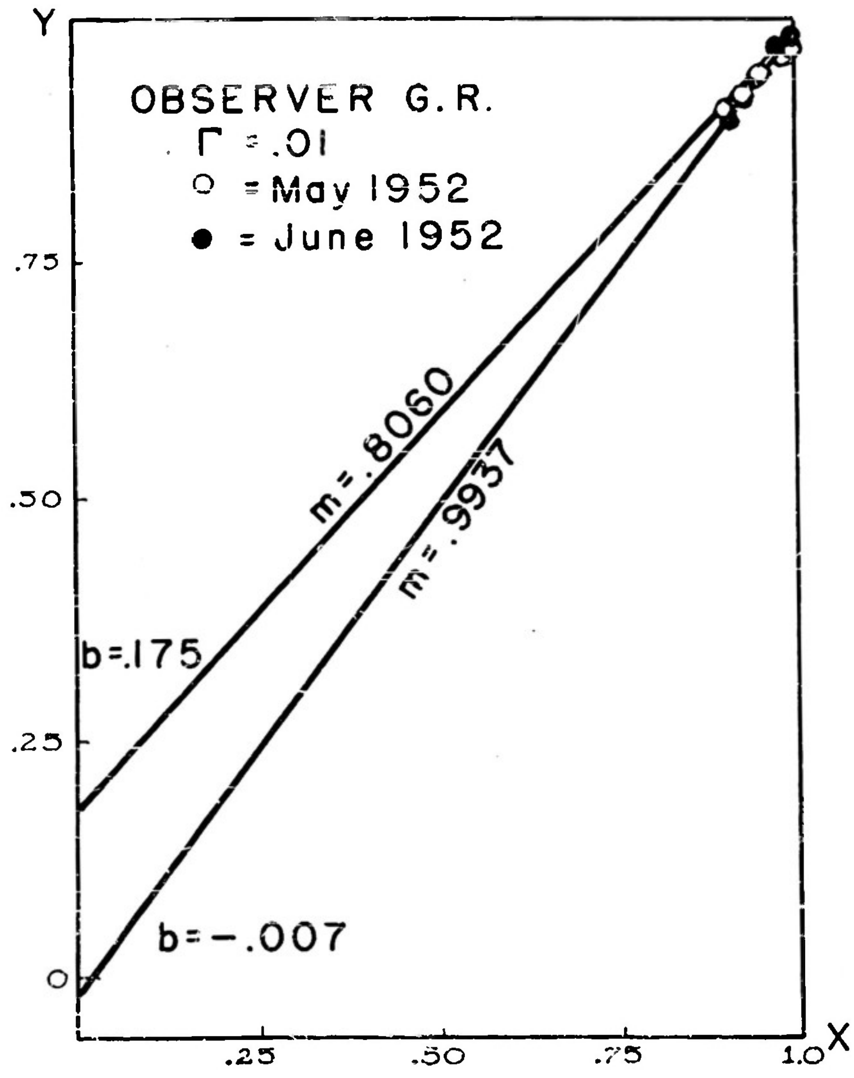


FIG. 25 THREE-POINT DVMC EXPERIMENT. PLOT EXHIBITING INSENSITIVITY OF EXPERIMENT IN DETERMINING m AND b.

Since the value of Y_0 at $X = 1$ ($\phi_2 = 0$) is found to be stable, the value of m is taken from the four-point experiment [see Part I, Section 5b (ii)] and b is determined from this value of m by the relation

$$b = Y_o - m. \tag{38}$$

3b (ii). The Four-Point Experiment. Four lights were presented to the observer. Two lights $Q_{+1} = (\gamma_1, \phi_1)$ and $Q_{-1} = (\gamma_1, \phi_{-1})$ were placed on the VMC $\gamma = \gamma_1$ symmetrically to the median. The other lights $Q_{+2} = (\gamma_2, \phi_2)$ and $Q_{-2} = (\gamma_2, \phi_{-2})$ were restricted to motion along a smaller VMC $\gamma = \gamma_2$ (see Fig. 26). The lights $Q_{\pm 2}$ on the inner circle were left fixed and the observer was asked to set the lights $Q_{\pm 1}$ on the outer circle according to the specific instructions:

''Four lights are presented to you. Two of them are fixed in position and the other two can be moved.'' (These lights and their range of motion are demonstrated.) ''Direct the experimenter to adjust these lights so

that the distance between them appears to you to be the same as the distance between the pair of fixed lights. Allow the eyes to roam freely both ways over the spatial interval between each pair of lights. Be sure to fixate on each light in turn and to sweep the eyes across the interval between each pair of lights until you are satisfied that the two distances appear to you to be the same.''

Settings of $Q_{+1} = (\gamma_1, \phi_1)$ and $Q_{-1} = (\gamma_1, \phi_{-1})$ were taken for five positions of Q_{+2} and and Q_{-2} on the VMC $\gamma = \gamma_2$ symmetric to the median with differences in azimuth $\Delta_2 = 10°, 30°, 50°$ (Fig. 26). At least three settings were taken at a given time for each position of $Q_{\pm 2}$. The entire series of experiments for the four values of Γ was performed twice.

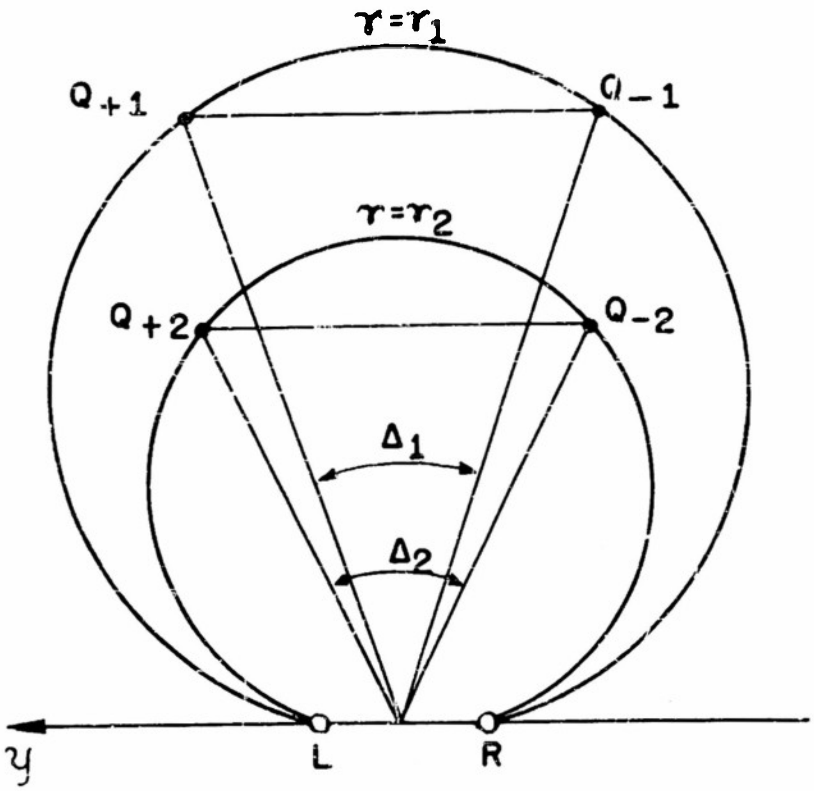


FIG. 26. PHYSICAL ARRANGEMENT IN FOUR-POINT DVMC EXPERIMENT.

The values of $Y = \sin \frac{1}{2}\Delta_1$ were determined for each of the values of $X = \sin \frac{1}{2}\Delta_2$ at a given value of Γ, and then Y was plotted against X. From equation (27)

$$m = Y/X$$

the slope m was computed from the data by weighting with respect to X,

$$m = \frac{\Sigma Y}{\Sigma X}. \tag{39}$$

The value of Y for each value of X is given in Table VII for each value of Γ, and the data are also plotted in Fig. 27.

Taking the value of m from (39) and the value of b from (38), the value of ω for $\Gamma = 0$ was obtained from formula (26)

$$\omega = \text{arc cosh} \frac{b}{[(1-b)^2 - m^2]^{\frac{1}{2}}}$$

The values of r were then computed according to formula (27)

$$r = \text{arc sinh } (m \sinh \omega).$$

The values r are given in Table VIII and plotted in Fig. 28 where they are compared with those obtained in other experiments.

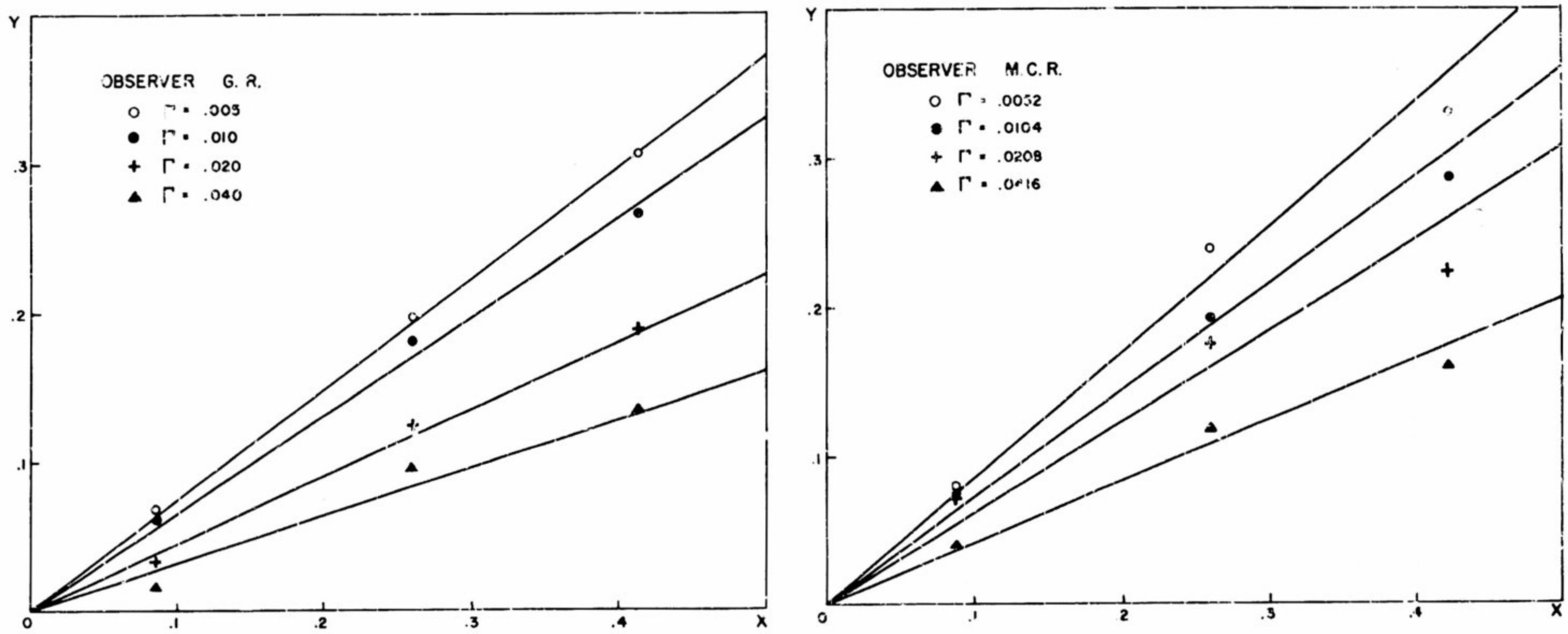


Fig. 27. Four-Point DVMC Experiment. Settings of Y ($\sin \frac{1}{2} \Delta_1$) for given values X ($\sin \frac{1}{2} \Delta_2$) and Γ.

TABLE VII

Four-Point Double Vieth-Müller Circle Experiment
Average values of $Y = \sin 1/2\,\Delta$ for given values of $X = \sin 1/2\,\Delta_2$ and Γ.

Observer G.R. $\gamma_1 = .025$

Average values of Y for given values of X and Γ

X \ Γ	.005	.01	.02	.04
.0872	.0685	.0604	.0334	.0160
.2588	.1972	.1805	.1249	.0962
.4226	.3050	.2660	.1887	.1353
m =	.7425	.6596	.4515	.3220

Observer M.C.R. $\gamma_1 = .026$

Average values of Y for given values of X and Γ

X \ Γ	.0052	.0104	.0208	.0416
.0872	.0809	.0749	.0732	.0390
.2588	.2389	.1923	.1755	.1176
.4226	.3291	.2858	.2231	.1586
m =	.8442	.7193	.6137	.4100

3c. The Equipartitioned Parallel Alleys

This experiment is that of 2b (ii). The mean values of Γ_1 and Γ_2, $\tan \phi_1$ and $\tan \phi_2$ for the four positions of γ_3 were taken from Table III and the values of r computed from these data. Hence the values of r given here represent sixteen experimental settings.

To compute the values of r we first determine the values of $S = \tan \phi_2 \,/\, \tan \phi_1$ and $T = \tan \phi_2 \,/\, \tan \phi_3$. For each of the experimental points r is then computed from formula (28) as

$$r_i = \text{arc tanh}\left[\frac{\tanh Y}{\sin \phi_i}\right]$$

TABLE VIII

Five values of r (Γ) computed from the results of the Double Vieth-Müller Circle Experiments.

Observer G.R.	γ_1 = .025	Observer M.C.R.	γ_1 = .026
Γ	r	Γ	r
.0000	1.48*	.0000	.95*
.0050	1.22	.0052	.83
.0100	1.13	.0104	.73
.0200	0.84	.0208	.63
.0400	0.63	.0416	.44

* Value of ω

where we use formula (30) for Y

$$\sinh Y = \tan \phi_2 \left[\frac{2 - (S + T)}{(S + T) - 2 ST} \right]^{1/2}$$

The results of these computations are given in Table IX. In Fig. 28 the results of this experimental series are compared with those obtained from the Blumenfeld Alleys and the Double Vieth-Müller Circle Experiments for the two observers who completed the full series of experiments.

3d. The Personal Characteristic, r (Γ).

If it be assumed that an individual's metrization of space is constant over long periods of life, then by determining the function r (Γ) we are able to give a useful and significant description of his space sense. This function r (Γ) may be thought of as a personal characteristic of the individual which describes his spatial responses for clueless vision in the same sense that his color matrix describes his responses to color mixtures.

With respect to the function r (Γ) we have sought to answer the following questions:

(a) What are its obvious characteristics?

(b) How well do the values obtained from different kinds of experiments agree with each other?

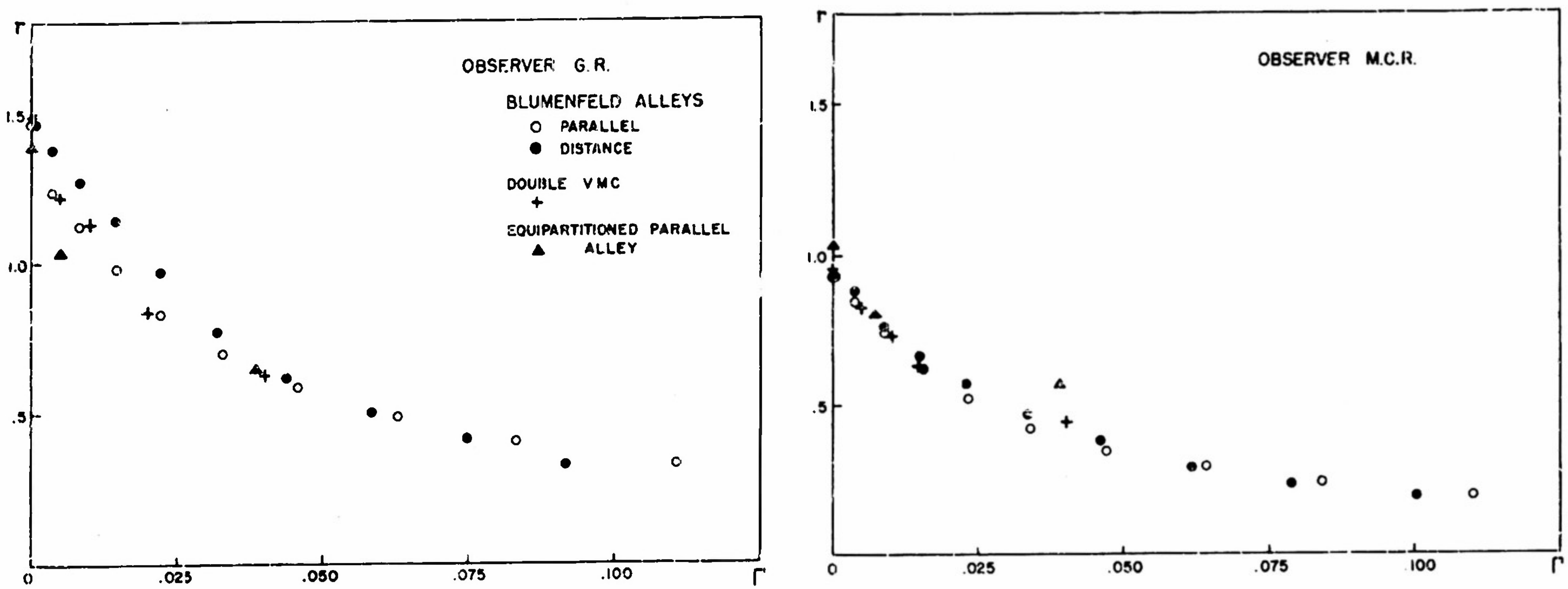


Fig. 28. The function r (Γ).

TABLE IX

Three values of r (Γ) computed from the results of the Equipartitioned Parallel Alley Experiment

Observer G. R. Γ	Observer G. R. r	Observer M.C.R. Γ	Observer M.C.R. r
.00000	1.3906*	.00000	1.0308*
.00562	1.0163	.00731	0.7993
.03904	0.6445	.03908	0.5676

* Value of ω

(c) Are there measurable differences between individuals?

A partial answer to these questions may be obtained from a perusal of Fig. 28 which summarizes the results of Tables V, VIII and IX.

(a) The function r (Γ) is a decreasing function and convex downward.

(b) The results of the three different experimental technics employed here do show a measure of agreement in the values of r (Γ). Whether or not the differences lie within the range of variability of the observer for any given experiment is a problem for further investigation and statistical analysis.

(c) Individual differences have been found. The function r (Γ) for observer G.R. is consistently larger than for observer M.C.R.

Another problem that has occupied our attention is that of determining a uniform way of interpolating a curve between the experimentally found values of the function, so that r (Γ) may be specified in terms of a limited number of real constants. To attempt a solution of this problem for all normal individuals is probably premature since it would require the testing of many more observers. Within the range of variability of the two observers employed here, however, the function r (Γ) can be represented adequately in the form

$$r = \frac{\omega}{1 + a\Gamma}$$

For observer G.R. we have approximately $\omega = 1.48$, $a = 33.2$; for observer M.C.R., $\omega = 1.00$, $a = 37.2$.

4. INSTRUMENTATION

In describing the geometry of binocular visual space we are concerned here not with thresholds and acuities, but with observations in the large where the eyes rove over extensive regions of space. For the present purpose we are interested in the response to gross stimuli rather than to the barely perceptible. The range of convergence and azimuth supplied by single light points in the laboratory is by no means an adequate domain for testing the range of the binocular responses to gross stimuli. Not only is it desirable to test with distant objects and at large angles of azimuth, but it is instructive to extend the range of observation as far as possible, even into the region of divergence. Such requirements can only be met by the use of special stereoscopic devices.

Since the use of a stereoscopic device of either of the types described here upsets the normal relationship between accommodation and convergence it is necessary to show that accommodation is a negligible or minor factor in binocular responses of the kind measured here. Campbell[16] in his study of size constancy phenomena in clueless vision was able to demonstrate that the substitution of stimuli formed by his stereoscopic device for single physical light points resulted in no appreciable change in the observer's settings. Further, he was able to commingle stimuli of the two kinds, again without appreciable change in the responses of the observer. We have used Campbell's demonstration as adequate justification for the use of our devices.*

4a. The Telestereoscope or ''Giant's Eyes'' Instrument.

This device is based on a mirror arrangement. A right-angled first surface mirror is placed symmetrically with respect to the median, apex toward observer (Fig. 29). Two mirrors are set symmetrically to the median so that the extension of the plane of each of the mirrors meets the extension away from the apex of the corresponding side of the right-angled mirror. The angle of intersection is denoted by α.

Let $\bar{Q}$ be a physical light viewed through the instrument. Let Q be the position of the

* ON THE OTHER HAND, WE HAVE NOT USED STIMULUS POINTS CLOSER THAN 50cm. SO THAT THE RANGE OF ACCOMMODATION IS NOT EXTREMELY LARGE.

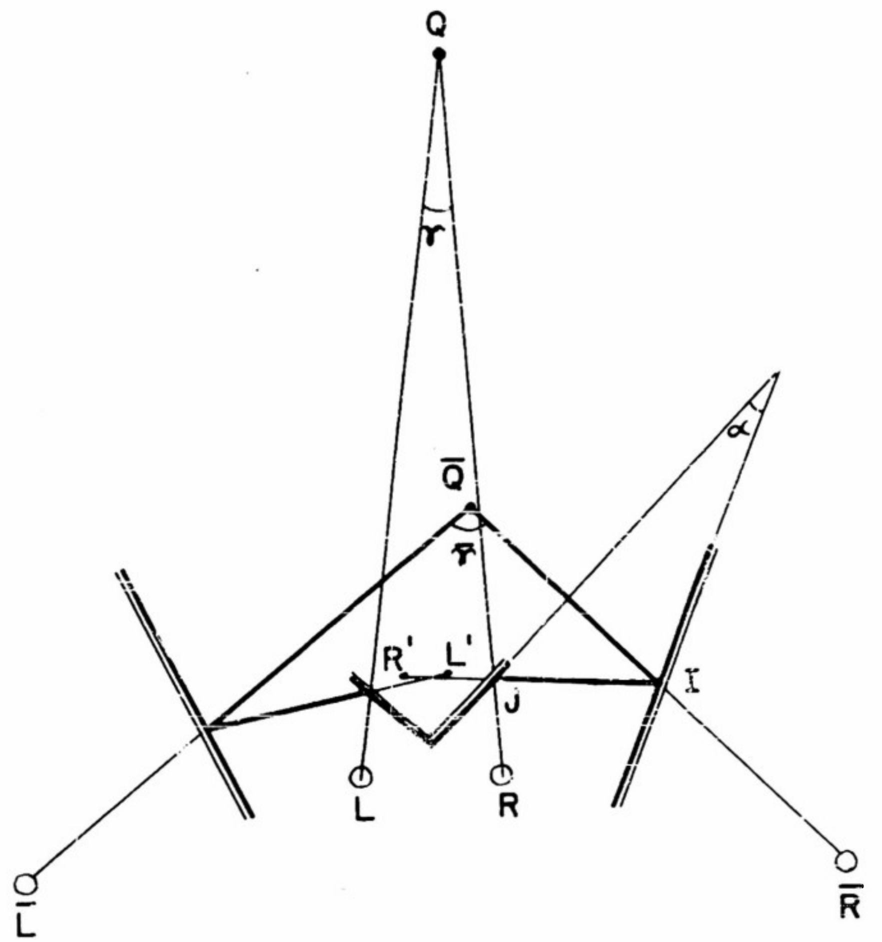


FIG. 29. THE TELESTEREOSCOPIC OR ''GIANT'S EYES'' INSTRUMENT.

binocular image of $\bar{Q}$ (i.e., the point at which the visual axes must cross to throw the separate images of $\bar{Q}$ on the respective foveae of the two eyes). Now consider the path of a ray of light from $\bar{Q}$ to the fovea of the right eye at R. Proceeding from R to Q we see first that the ray which reaches R from its side of the 90° mirror must have been reflected from a ray directed through the image R′ of R in the mirror. Similarly, the ray directed through R′ must have been reflected at the side mirror from a ray directed through the image $\bar{R}$ of R′ in that mirror. By symmetry we determine the corresponding points L′ and $\bar{L}$ for the left eye.

Thus, to determine the position of Q from that of $\bar{Q}$ we trace the rays to the two eyes and extend the terminal segments at the eyes to their point of intersection. Thus, for the right eye, we draw $\bar{Q}\,\bar{R}$ and find its intersection I with the side mirror. We then draw IR′ and determine the intersection J with the right face of the 90° mirror. The point Q will then lie on the extension of JR.

For algebraic convenience we use bipolar coordinates $\bar{\gamma}$, $\bar{\phi}$ with respect to the points $\bar{L}$, $\bar{R}$ (the "Giant's Eyes") to specify the position of $\bar{Q}$, and ordinary bipolar coordinates γ, ϕ to give the position of Q. These coordinates are related to each other by the equations

$$\gamma = \bar{\gamma} - 4\alpha$$
$$\phi = \bar{\phi}.$$

The "Giant's Eyes" instrument was used in the Double VMC experiments of Section 2b (i).

4b. The Polaroid Rack

In this instrument two lights Q_L and Q_R are presented separately by means of polaroids to the left and right eyes, respectively. This is interpreted in the same way as a physical light placed at the point Q where the visual axes cross when the right eye fixes Q_R and the left eye fixes Q_L (cf. Campbell[16]). The rack consists of a set of bars placed on lines x = constant. Each pair of lights Q_R and Q_L which are to be fused in the above manner is placed on a block so that the two may slide along a bar together as a unit. The separation of the two lights of a pair is adjustable by means of a thumbscrew. Polaroids are mounted before the lights and the eyes so that each pair of lights gives only one image to each eye (Fig. 30).

Let the cartesian coordinates for the two lights of a pair be given by $Q_R = (X, Y_R)$, $Q_L = (X, Y_L)$. The line OQ passes through the point $Q^* = (X, Y)$ half way between Q_R and Q_L where

$$Y = \tfrac{1}{2}(Y_R + Y_L).$$

The separation of Q_R and Q_L is

$$S = Y_R - Y_L$$

The cartesian coordinates (x, y) of Q are given by

$$x = \frac{pX}{p + S} \tag{40}$$

$$y = \frac{pY}{p + S}$$

where p is the interpupillary distance of the observer. From equation (40) approximate formulas

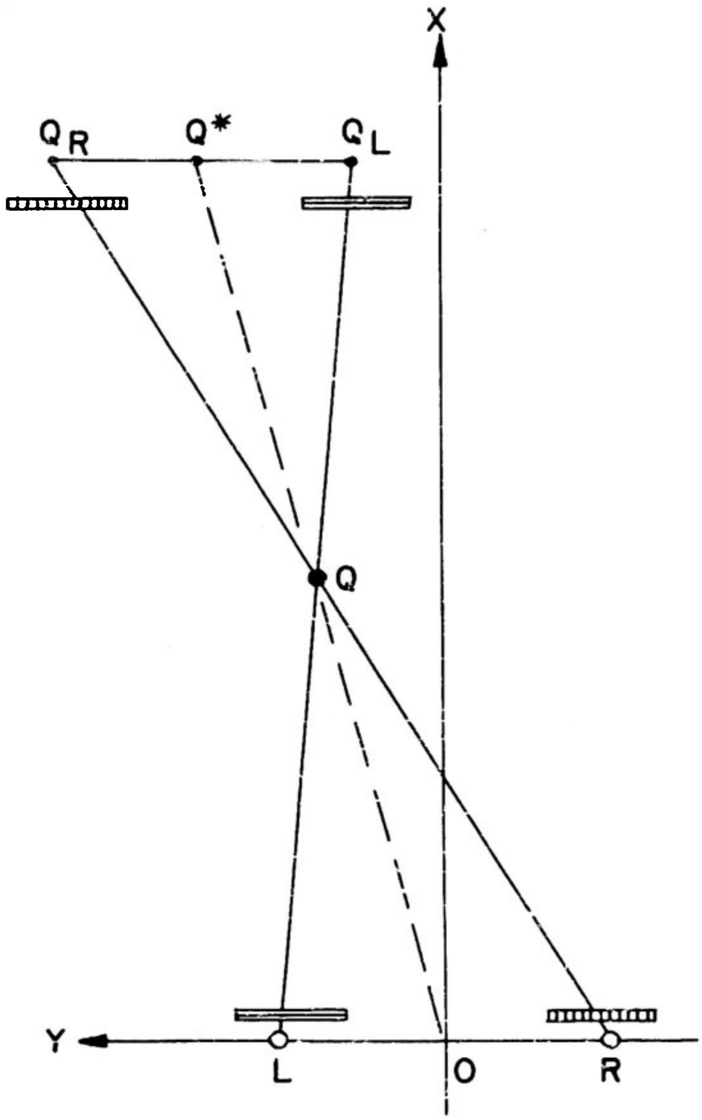


FIG. 30. SCHEMATIC REPRESENTATION OF THE PRINCIPLE OF THE POLAROID RACK.

for the bipolar coordinates $(\gamma\ ,\ \phi)$ of Q are easily obtained

$$\tan \phi = Y/X$$

$$\gamma = \frac{(p + S)\cos^2 \phi}{X}$$

The polaroid rack was used in the settings of the equipartitioned parallel allevs Sections 2b (ii) and 3c.

No account was taken of the slight errors due to a 2.5 mm. thickness of glass covering the polaroids before the eyes. Similarly, although all observers with the exception of A.A.B. wore glasses, the deviations due to refracting elements were not taken into account. To avoid entering into such considerations it would be desirable to employ emmetropic observers.

5. CONCLUSIONS

The experimental work presented here is direct evidence that Luneberg's approach to binocular vision is at once sensible and feasible. In fact this evidence strongly supports Luneburg's major conclusions:

(a) The binocular visual space is a determinate metric space with constant characteristics for a given observer.

(b) By experiment, it is possible to determine the metric of an observer's visual space and so to completely characterize the geometry of his binocular visual sense.

(c) The metric is that of riemannian space of constant negative curvature, the so-called hyperbolic space.

In particular, the experiments show that in all likelihood, the metric of visual space may be written in terms of special coordinates attached to the stimulus configuration; i.e., the iseikonic coordinates, Γ, Φ, Θ. The problem of determining the metric for a given observer is then reduced to the problem of determining the one function $r(\Gamma)$.

The function $r(\Gamma)$ is to be thought of as a sensory characteristic of the observer which describes his geometric visual sense, much as an individual's color matrix describes his sense of hue and saturation. The determination of norms for the function might therefore be useful, especially with regard to our understanding of deviant or abnormal binocular function. As yet it is too early to make predictions concerning the eventual usefulness (clinically or otherwise) of the theory. A great deal of work to set up standards must first be undertaken. Yet some practical results will undoubtedly follow from the increased understanding we already have. For example, it may be suggested that parallel rows of guide lights be set up along all airplane runways at a uniform standard separation of the rows and at a uniform standard spacing of the lights, so that a pilot landing at night can rely on facing the same situation each time he lands at any field and on any runway. If this were carried into national or international standard patterns, it might do much to reduce hazards of visual landings - particularly at strange airports.

Perhaps similar standards would prove useful in other applications where space judgments must be made in a situation providing reduced clues; e.g., it might lead to a consideration and solution of the problem of integrating the magnification of all binocular viewing instruments. If the interpupillary distance of a 6X binocular, for example, were optically magnified by the same factor of 6, the relative changes in convergence required in using the instrument should lead to a more realistic appraisal, by the observer, of frontal distances involved in the field of view.

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